Stability analysis of the value function and its maximizers set via variational convergence notions

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Abstract

We investigate the stability of the value function and the set of its maximizers for a parametric optimization problem according to Berge's maximum theorem. To accomplish this, we use variational convergence notions for perturbing both the function and the multifunction. Our findings are applied to generalized Nash equilibrium problems and to finite-horizon dynamic programming models.

Keywords: Berge's maximum theorem; Variational convergence; Finite-horizon dynamic programming; Generalized Nash games.

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1 Introduction

The classical Berge's maximum theorem [9] holds significant relevance in fields such as economic theory, optimal control, and optimization theory. For instance, in demand theory, it is concerned with an agent's optimal consumption concerning prices and income, while in capital theory, with the

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optimal investment strategy based on the existing capital stock. Berge's maximum theorem plays a pivotal role in addressing these issues, as it confirms the continuity of the value function and the upper semicontinuity of the optimal choice. The former property has been used in existence theorems and characterization results for dynamic programming, whereas the latter one has been used in applying fixed point theorems of Kakutani type.

The stability of optimization problems is of paramount importance due to the inherent inaccuracy of such problems. This inaccuracy stems from various sources, including predictive errors in forecasting certain data inputs (such as future demands or returns), measurement errors in empirical data (such as parameters of devices or processes), implementation errors during computation (including approximations and rounding), and system errors in mathematical modeling. As a result, the solution obtained is, at best, an approximation to the true solution. In order for this approximate solution to be practically useful, decision-makers need access to stability information about the problem that depends on variations of the data.

For the parametric optimization problem according to Berge's maximum theorem, the stability analysis consists of studying the convergence of the value functions and the solution set multifunctions, when the data (objective function and feasible multifunction) are subject to variations via variational convergence notions. Zolezzi [37] studied the stability of the value function at fixed parameter. Lignola and Morgan [22] studied the stability of the value function and of the solution multifunction w.r.t. perturbations of the objective function and of the feasible multifunction. Our results complement those in [22, 37] and shed new light on them since we use other variational convergence notions: continuous or hypo-convergences for functions, together with continuous or graphical convergences for multifunctions. These convergence notions are described in terms of Painlevé-Kuratowski convergence of sets (images, hypographs, graphs). Some references on these convergence notions appear in the books of Attouch [4], Beer [8], Burachik and Iusem [12], Hu and Papageorgiou [19], and Rockafellar and Wets [33].

We achieve limsup convergence for the solution multifunction without altering its original definition. However, to attain its liminf convergence, we find it necessary to extend the definition of the solution set. Specifically, we consider the concept of an ε-approximate multifunction solution, and through this approach, we achieve liminf convergence for the solution multifunction. This method has been used in [1, 24, 25, 26, 29, 37] to study diverse optimization problems, equilibrium problems and minmax problems.

As Berge's maximum theorem holds significant relevance in generalized Nash games, we leverage our findings to examine the approximation of generalized Nash equilibria. We obtain conditions for the convergence of the approximate generalized Nash equilibria via a direct approach. Contrary to optimization theory, there are limited studies addressing the approximation of generalized Nash equilibria. To the best of our knowledge, the works [17, 29] primarily concentrate on the classical Nash game. As another application of our results, we perturb a finite-horizon dynamic programming model by a sequence of plans and multifunctions representing the feasible sets of plans. We prove the stability of the model under natural assumptions.

The paper is organized as follows. In Section 2, we introduce the notation and recall some preliminaries. Section 3 is devoted to present our main results. In Section 4, we apply our results to generalized Nash equilibrium problems and to finite-horizon dynamic programming models.

2 Notation and preliminaries

We are interested in studying the stability of the parametric optimization problem according to Berge's maximum theorem:

$$v(y) := \sup_{x\in\Phi(y)} u(x,y). \qquad (\mathcal{P}_y)$$

where $y \in \mathbb{R}^m$ is a parameter vector, $u\colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is an extended-valued objective function and $\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a feasible multifunction. The function $v\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ is called the value function. The associated solution multifunction $S\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is defined by

$$S(y) := \operatorname*{arg\,max}_{x\in\Phi(y)} u(x,y) = \{x \in \Phi(y)\colon v(y) = u(x,y)\}.$$

When $\Phi(y) = \emptyset$, we have $v(y) = -\infty$ and $S(y) = \emptyset$.

We denote by $x = (x_1,\dots,x_n)$ a vector from $\mathbb{R}^n$, by $y = (y_1,\dots,y_m)$ a vector from $\mathbb{R}^m$, by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$, by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ the extended set of real numbers, by $\varepsilon_k \searrow 0$ when $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for all k, and by $\varepsilon_k \downarrow 0$ when $\varepsilon_k \to 0$ with $0 < \varepsilon_{k+1} < \varepsilon_k$ for all k. For an extended-valued function $f\colon \mathbb{R}^\ell \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}_U f = \{x \in \mathbb{R}^\ell\colon f(x) > -\infty\}$ its U-domain; by $\mathrm{dom}_L f = \{x \in \mathbb{R}^\ell\colon f(x) < +\infty\}$ its L-domain; by $\mathrm{epi}\, f := \{(x,\lambda) \in \mathbb{R}^{\ell+1}\colon f(x) \le \lambda\}$ its epigraph; and by $\mathrm{hyp}\, f := \{(x,\lambda) \in \mathbb{R}^{\ell+1}\colon \lambda \le f(x)\}$ its hypograph. We say that f is U-proper (resp. L-proper) if $f(x) < +\infty$ (resp. $f(x) > -\infty$) for all $x \in \mathbb{R}^\ell$ and $\mathrm{dom}_U f \ne \emptyset$ (resp. $\mathrm{dom}_L f \ne \emptyset$).

To perturb the data of the parametric optimization problem, we recall set convergence notions from [33]. For a sequence of sets $\{C_k\}$ from $\mathbb{R}^\ell$, $\limsup_k C_k := \{x \in \mathbb{R}^\ell\colon \exists\, x^{k_j} \in C_{k_j} \text{ s.t. } x^{k_j} \to x\}$ is its outer limit and $\liminf_k C_k := \{x \in \mathbb{R}^\ell\colon \exists\, x^k \in C_k \text{ s.t. } x^k \to x\}$ is its inner limit, where $\{x^{k_j}\}$ is a subsequence of $\{x^k\}$. We say that $\{C_k\}$ converges in the sense of Painlevé-Kuratowski to C, denoted by $C_k \to C$ or $\lim_k C_k = C$, if $\limsup_k C_k = C = \liminf_k C_k$, or equivalently if $\limsup_k C_k \subset C \subset \liminf_k C_k$.

A sequence of sets $\{C_k\}$ from $\mathbb{R}^\ell$ is said to be: eventually bounded if there exists N such that $\bigcup_{k\ge N} C_k$ is bounded; strongly eventually bounded if it is eventually bounded and $\bigcup_{k\ge j} C_k$ is nonempty for all $j \in \mathbb{N}$; nonempty-valued if C_k is nonempty for all $k \in \mathbb{N}$; and eventually nonempty-valued (resp. empty-valued) if there exists N such that $C_k \ne \emptyset$ (resp. $C_k = \emptyset$) for all $k \ge N$. Clearly, nonempty-valuedness implies eventually nonempty-valuedness which in turn implies that $\bigcup_{k\ge j} C_k \ne \emptyset$ for all

$j \in \mathbb{N}$. If $\{C_k\}$ is eventually bounded and eventually nonempty-valued, then it is strongly eventually bounded. The following conditions are equivalent:

- $\bigcup_{k \geq j} C_k \neq \emptyset$, for all $j \in \mathbb{N}$;
- $\{C_k\}$ is not eventually empty-valued;
- There exists a subsequence $\{C_{k_j}\}$ of $\{C_k\}$ such that $C_{k_j} \neq \emptyset$, for all $j \in \mathbb{N}$.

We establish conditions under which $\limsup_k C_k$ is nonempty and compact. To do this, we recall the "escaping to the horizon" property: $C_k \to \emptyset$ (or equivalently $\limsup_k C_k = \emptyset$). By [33, Corollary 4.11], the following equivalences hold:

- $\limsup_k C_k = \emptyset$;
- For every $\rho > 0$ there exists $N \in \mathbb{N}$ such that $C_k \cap \rho\mathbb{B} = \emptyset$, for all $k \geq N$;
- $d_{C_k}(0) \to +\infty$.

From this, we have $\limsup_k C_k \neq \emptyset$ iff there exists $\rho > 0$ such that $\bigcup_{k \geq j}(C_k \cap \rho\mathbb{B}) \neq \emptyset$, for all $j \in \mathbb{N}$.

Proposition 1. *If $\{C_k\}$ is strongly eventually bounded, then*

$$\limsup_k C_k \text{ is nonempty and compact.} \tag{1}$$

Moreover, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\bigcup_{k \geq N} C_k \subset \{x \in \mathbb{R}^\ell \colon d_{\limsup_k C_k}(x) < \varepsilon\}. \tag{2}$$

Proof. We prove the first part. Let us denote $A_j := \bigcup_{k \geq j} C_k$ for $j \in \mathbb{N}$. By hypothesis, A_j is nonempty for all $j \in \mathbb{N}$ and bounded for j large enough. Hence $\operatorname{cl} A_j$ is nonempty and compact for j large enough. As $\operatorname{cl} A_{j+1} \subset \operatorname{cl} A_j$ for all $j \in \mathbb{N}$, by Cantor's intersection theorem, we infer that $\bigcap_j \operatorname{cl} A_j$ is nonempty and compact. The result follows since $\bigcap_j \operatorname{cl} A_j = \limsup_k C_k$ (see [33, Exercise 4.2(b)]).

We prove the second part. Let us denote $C_\varepsilon := \{x \in \mathbb{R}^\ell \colon d_{\limsup_n C_n}(x) < \varepsilon\}$. As $\bigcap_j \operatorname{cl} A_j \subset C_\varepsilon$, we have $(\bigcap_j \operatorname{cl} A_j) \cap C_\varepsilon^{\mathrm{c}} = \emptyset$; i.e., $\bigcap_j (\operatorname{cl} A_j \cap C_\varepsilon^{\mathrm{c}}) = \emptyset$ with the sets in parentheses being closed. From this and Cantor's intersection theorem, we deduce that there exists N such that $\operatorname{cl} A_N \cap C_\varepsilon^{\mathrm{c}} = \emptyset$; i.e., $\bigcup_{k \geq N} C_k \subset C_\epsilon$. □

Remark 2. 1. *Condition* (1) *does not imply that $\{C_k\}$ is eventually bounded. Indeed, for $C_k = [0,1] \cup \{k\}$ for all k, we have $\lim_k C_k = [0,1]$ but $\{C_k\}$ is not eventually bounded.*

2. *Condition* (2) *does not imply that $\{C_k\}$ is eventually bounded. Indeed, this is shown by taking $C_k = \mathbb{R}^\ell$ for all k.*

We recall some notions of set-valued analysis from [12, 33]. A multifunction or set-valued mapping $\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a mapping that associates to any vector y in $\mathbb{R}^m$ a set $\Phi(y)$ in $\mathbb{R}^n$. We denote by $\operatorname{dom}\Phi := \{y \in \mathbb{R}^m \colon \Phi(y) \neq \emptyset\}$ its domain and by $\operatorname{gph}\Phi := \{(y,x) \in \mathbb{R}^{m+n} \colon x \in \Phi(y)\}$ its graph. A mapping Φ is said to be proper if it has a nonempty domain. We say that Φ is: outer semicontinuous (osc) at y if $\limsup_k \Phi(y^k) \subset \Phi(y)$ for all $y^k \to y$; inner semicontinuous (isc) at y if $\Phi(y) \subset \liminf_k \Phi(y^k)$ for all $y^k \to y$; upper semicontinuous (usc) at y if for any open set V containing $\Phi(y)$ there is a neighborhood U of y such that $\Phi(U) \subset V$; lower semicontinuous (lsc) at y if for any open set V with $\Phi(y) \cap V \neq \emptyset$ there is a neighborhood U of y such that $\Phi(z) \cap V \neq \emptyset$ for every $z \in U$; continuous at y if it is osc and isc at y; K-continuous at y if it is usc and lsc at y; locally bounded at y if for some neighborhood V of y the set $\Phi(V)$ is bounded; and osc (respectively, isc, usc, lsc, continuous, K-continuous, locally bounded) if it is so at every y. We say that Φ is N-valued if $\Phi(y)$ has property N for every y (e.g. closed-valued); and uniformly bounded if $\Phi(\mathbb{R}^m)$ is bounded. For $\Phi, \Psi \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we write $\Phi \subset \Psi$ if $\Phi(y) \subset \Psi(y)$ for all y and $\Phi = \Psi$ if $\Phi \subset \Psi$ and $\Psi \subset \Phi$.

We list some continuity properties for multufunctions to be used later on.

Proposition 3. ([5, 10, 12, 33])

(a) *Φ is osc iff* $\operatorname{gph}\Phi$ *is closed. If Φ is osc, then Φ is closed-valued. Moreover, Φ is isc iff Φ is lsc.*

(b) *If Φ is usc at y and $\Phi(y)$ is closed, then Φ is osc at y. The reverse implication holds, if in addition Φ is locally bounded at y.*

(c) *Φ is locally bounded iff $\Phi(B)$ is bounded for every bounded set B iff whenever $x^k \in \Phi(y^k)$ and $\{y^k\}$ is bounded, the sequence $\{x^k\}$ is bounded.*

(d) *If Φ is usc and compact-valued, then Φ is locally bounded and osc. In addition, if $\Phi(\mathbb{R}^m)$ is compact, then Φ is usc and compact-valued iff Φ is osc.*

> Assumption: From now on, we assume that $u \colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is an extended-valued function and $\Phi \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a proper multifunction.

We recall the next result that will be deduced below from our results. Part (c) is referred to as "Berge's maximum theorem".

Proposition 4. [19, Propositions 3.1–3.4]

(a) *If $u \colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is lsc and $\Phi \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is lsc, then $v \colon \mathbb{R}^m \to \overline{\mathbb{R}}$ is lsc.*

(b) *If $u \colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is usc and $\Phi \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is nonempty-valued, compact-valued and usc, then $v \colon \mathbb{R}^m \to \overline{\mathbb{R}}$ is usc.*

(c) *If $u \colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is continuous and $\Phi \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is nonempty-valued, compact-valued and K-continuous, then $v \colon \mathbb{R}^m \to \overline{\mathbb{R}}$ is continuous and $S \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is nonempty-valued, compact-valued and usc.*

Remark 5. 1. *Concerning part (a), in* [19, Proposition 3.1] *it is additionally assumed that* Φ *is nonempty-valued. This assumption can be dropped. Indeed, let* y *and* $y^k \to y$ *be fixed. If* $\Phi(y) = \emptyset$, *then* $v(y) = -\infty \le \liminf_k v(y^k)$. *On the other hand, if* $\Phi(y) \neq \emptyset$, *then for every* $\varepsilon > 0$ *there exists* $x_\varepsilon \in \Phi(y)$ *such that* $v(y) - \varepsilon < u(x_\varepsilon, y)$. *As* Φ *is isc, we have* $x_\varepsilon \in \liminf_k \Phi(y^k)$ *and thus there exists* $x^k \in \Phi(y^k) \to x_\varepsilon$. *By hypothesis,* $v(y) - \varepsilon < u(x_\varepsilon, y) \le \liminf_k u(x^k, y^k) \le \liminf_k v(y^k)$. *As* $\varepsilon > 0$ *was arbitrary, we infer that* $v(y) \le \liminf_k v(y^k)$. *Hence* v *is lsc.*

2. *By Proposition 3,* Φ *is usc and compact-valued iff* Φ *is osc and locally bounded. This allows us to reformulate Berge's maximum theorem as follows: "If* $u\colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ *is continuous and* $\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is nonempty-valued, locally bounded and continuous, then* $v\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *is continuous and* $S\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is nonempty-valued, locally bounded and osc".*

We study the behavior of the value function and the solution multifunction

$$v(y) := \sup_{x \in \Phi(y)} u(x, y) \quad \text{and} \quad S(y) = \{x \in \Phi(y) \colon v(y) = u(x, y)\}, \tag{3}$$

when the data u and Φ are subject to variation. To do this, we approximate u by $\{u^k\}$ and Φ by $\{\Phi^k\}$ by using variational convergence notions. We denote the respective approximations of the value function v by $\{v^k\}$ and of the solution multifunction S by $\{S^k\}$; i.e.,

$$v^k(y) := \sup_{x \in \Phi^k(y)} u^k(x, y) \quad \text{and} \quad S^k(y) := \{x \in \Phi^k(y) \colon v^k(y) = u^k(x, y)\}. \tag{4}$$

We will establish convergence results for the sequences $\{v^k\}$ and $\{S^k\}$. To do this, we first recall the notion of hypo-convergence to approximate functions.

Definition 6. [33] *For a sequence of extended-valued functions* $\{f^k \colon \mathbb{R}^\ell \to \overline{\mathbb{R}}\}$, *its lower hypo-limit is the function* $h\text{-}\liminf_k f^k \colon \mathbb{R}^\ell \to \overline{\mathbb{R}}$ *having as its hypograph*

$$\operatorname{hyp}(h\text{-}\liminf_k f^k) = \liminf_k (\operatorname{hyp} f^k),$$

and its upper hypo-limit is the function $h\text{-}\limsup_k f^k \colon \mathbb{R}^\ell \to \overline{\mathbb{R}}$ *having as its hypograph*

$$\operatorname{hyp}(h\text{-}\limsup_k f^k) = \limsup_k (\operatorname{hyp} f^k).$$

We have $h\text{-}\liminf_k f^k \le h\text{-}\limsup_k f^k$ *and when these limits are equal to* f, *we say that the hypo-limit* $h\text{-}\lim_k f^k$ *exists and that the sequence hypo-converges to* f, *denoted by* $h\text{-}\lim_k f^k = f$ *or* $f^k \xrightarrow{h} f$.

Remark 7. 1. *Clearly,* $f \le h\text{-}\liminf_k f^k$ *iff* $\operatorname{hyp} f \subset \liminf_k (\operatorname{hyp} f^k)$ *and* $h\text{-}\limsup_k f^k \le f$ *iff* $\limsup_k (\operatorname{hyp} f^k) \subset \operatorname{hyp} f$. *Hence the following conditions are equivalent:*

- $f^k \xrightarrow{h} f$;
- $\operatorname{hyp} f^k \to \operatorname{hyp} f$;
- $f \le h\text{-}\liminf_k f^k$ *and* $h\text{-}\limsup_k f^k \le f$.

We have the following formulas for hypo-limits at every $x \in \mathbb{R}^\ell$ *(see* [33, Proposition 7.2]*):*

$$\begin{aligned} (h\text{-}\liminf_k f^k)(x) &= \max\{\alpha \in \overline{\mathbb{R}} \colon \exists x^k \to x \textit{ with } \liminf_k f^k(x^k) = \alpha\}, \\ (h\text{-}\limsup_k f^k)(x) &= \max\{\alpha \in \overline{\mathbb{R}} \colon \exists x^k \to x \textit{ with } \limsup_k f^k(x^k) = \alpha\}. \end{aligned}$$

Hence

$$f \le h\text{-}\liminf_k f^k \iff \forall x \in \mathbb{R}^\ell, \exists x^k \to x \colon f(x) \le \liminf_k f^k(x^k), \tag{5}$$

$$h\text{-}\limsup_k f^k \le f \iff \forall x \in \mathbb{R}^\ell, \forall x^k \to x \colon \limsup_k f^k(x^k) \le f(x). \tag{6}$$

Clearly, if $f^k \xrightarrow{h} f$*, then for every* $x \in \mathbb{R}^\ell$ *there exists* $x^k \to x$ *such that* $f^k(x^k) \to f(x)$*.*

2. *If* $f^k \xrightarrow{h} f$*, then* f *is usc* (see [33, Proposition 7.4(a) and p. 243])*. In particular, if* $f^k \equiv f$*, then, in order to obtain* $f^k \xrightarrow{h} f$*, the function* f *must be usc.*

3. [33] *For a sequence of extended-valued functions* $\{f^k \colon \mathbb{R}^\ell \to \overline{\mathbb{R}}\}$*, its lower and upper epi-limits are defined by*

$$\operatorname{epi}(e\text{-}\liminf_k f^k) = \limsup_k(\operatorname{epi} f^k) \quad \textit{and} \quad \operatorname{epi}(e\text{-}\limsup_k f^k) = \liminf_k(\operatorname{epi} f^k).$$

We say that the epi-limit $e\text{-}\lim_k f^k$ *exists and that the sequence epi-converges to* f*, denoted by* $e\text{-}\lim_k f^k = f$ *or* $f^k \xrightarrow{e} f$*, if* $e\text{-}\liminf_k f^k = e\text{-}\limsup_k f^k = f$*. Clearly,* $e\text{-}\limsup_k f^k \le f$ *iff* $\operatorname{epi} f \subset \liminf_k(\operatorname{epi} f^k)$ *and* $f \le e\text{-}\liminf_k f^k$ *iff* $\limsup_k(\operatorname{epi} f^k) \subset \operatorname{epi} f$*. Hence* $f^k \xrightarrow{e} f$ *iff* $\operatorname{epi} f^k \to \operatorname{epi} f$*. We have the following formulas for epi-limits at every* $x \in \mathbb{R}^\ell$*:*

$$\begin{aligned} (e\text{-}\liminf_k f^k)(x) &= \min\{\alpha \in \overline{\mathbb{R}} \colon \exists x^k \to x \textit{ with } \liminf_k f^k(x^k) = \alpha\}, \\ (e\text{-}\limsup_k f^k)(x) &= \min\{\alpha \in \overline{\mathbb{R}} \colon \exists x^k \to x \textit{ with } \limsup_k f^k(x^k) = \alpha\}. \end{aligned}$$

Hence

$$f \le e\text{-}\liminf_k f^k \iff \forall x \in \mathbb{R}^\ell, \forall x^k \to x \colon f(x) \le \liminf_k f^k(x^k), \tag{7}$$

$$e\text{-}\limsup_k f^k \le f \iff \forall x \in \mathbb{R}^\ell, \exists x^k \to x \colon \limsup_k f^k(x^k) \le f(x). \tag{8}$$

Clearly,

$$\begin{aligned} f \le h\text{-}\liminf_k f^k &\iff e\text{-}\limsup_k(-f^k) \le -f, \\ h\text{-}\limsup_k f^k \le f &\iff -f \le e\text{-}\liminf_k(-f^k). \end{aligned}$$

Hence $f^k \xrightarrow{h} f$ *iff* $-f^k \xrightarrow{e} -f$*.*

4. *We have* $\operatorname{hyp} f \ne \emptyset$ *when* $\operatorname{dom}_U f \ne \emptyset$*. In addition, if* f *is* U*-proper, then* $(x, \lambda) \in \operatorname{hyp} f$ *for every* $x \in \operatorname{dom}_U f$ *and* $\lambda \in \mathbb{R}$ *such that* $\lambda \le f(x)$*. In particular,* $(x, f(x)) \in \operatorname{hyp} f$*. Accordingly, we relate* $\operatorname{epi} f$ *and* L*-properness.*

We recall the notion of continuous convergence to approximate functions.

Definition 8. ([33]) *A sequence of extended-valued functions* $\{f^k\colon \mathbb{R}^\ell \to \overline{\mathbb{R}}\}$ *is said to convergence continuously to* f*, denoted by* $f^k \xrightarrow{c} f$*, if* $f^k(x^k) \to f(x)$*, for every* $x^k \to x$*.*

To develop our approach, we define lower and upper continuous limits that split continuous convergence into two parts.

Definition 9. *For a sequence of extended-valued functions* $\{f^k\colon \mathbb{R}^\ell \to \overline{\mathbb{R}}\}$*, its lower continuous limit is the function* $c\text{-}\liminf_k f^k\colon \mathbb{R}^\ell \to \overline{\mathbb{R}}$*, defined by*

$$(c\text{-}\liminf_k f^k)(x) := \min\{\alpha \in \overline{\mathbb{R}}\colon \exists x^k \to x \text{ with } \liminf_k f^k(x^k) = \alpha\},$$

and its upper continuous limit is the function $c\text{-}\limsup_k f^k\colon \mathbb{R}^\ell \to \overline{\mathbb{R}}$*, defined by*

$$(c\text{-}\limsup_k f^k)(x) := \max\{\alpha \in \overline{\mathbb{R}}\colon \exists x^k \to x \text{ with } \limsup_k f^k(x^k) = \alpha\}.$$

Clearly, $c\text{-}\liminf_k f^k \le c\text{-}\limsup_k f^k$ *and when these limits are equal to* f*, we say that the continuous limit* $c\text{-}\lim_k f^k$ *exists and it is equal to* f*. In this case, we write* $c\text{-}\lim_k f^k = f$*.*

Remark 10. 1. *By Definition 9 and Remark 7, we have*

$$c\text{-}\liminf_k f^k = e\text{-}\liminf_k f^k \quad \text{and} \quad c\text{-}\limsup_k f^k = h\text{-}\limsup_k f^k.$$

From this and (6)–(7)*, we have*

$$f \le c\text{-}\liminf_k f^k \iff \forall x \in \mathbb{R}^\ell, \forall x^k \to x\colon f(x) \le \liminf_k f^k(x^k), \tag{9}$$

$$c\text{-}\limsup_k f^k \le f \iff \forall x \in \mathbb{R}^\ell, \forall x^k \to x\colon \limsup_k f^k(x^k) \le f(x). \tag{10}$$

It is easy to check that the following conditions are equivalent:

- $f^k \xrightarrow{c} f$*;*
- $c\text{-}\lim_k f^k = f$*;*
- $f \le c\text{-}\liminf_k f^k$ *and* $c\text{-}\limsup_k f^k \le f$*.*

From this and above equalities, we obtain that $f^k \xrightarrow{c} f$ *iff* $f \le e\text{-}\liminf_k f^k$ *and* $h\text{-}\limsup_k f^k \le f$*. Lignola and Morgan* [22] *defined continuous convergence by using this equivalence.*

2. *If* $f^k \xrightarrow{c} f$*, then* f *is continuous* (*see* [33, Theorem 7.14])*. In particular, if* $f^k \equiv f$*, then to have* $f^k \xrightarrow{c} f$*, the function* f *must be continuous.*

3. *By conditions* (9)–(10) *and* (5)–(6) *and item (1), we have*

$$\begin{aligned} f \le c\text{-}\liminf_k f^k &\iff f \le e\text{-}\liminf_k f^k \implies f \le h\text{-}\liminf_k f^k,\\ c\text{-}\limsup_k f^k \le f &\iff h\text{-}\limsup_k f^k \le f \implies e\text{-}\limsup_k f^k \le f. \end{aligned}$$

From this, we infer that $f^k \xrightarrow{c} f$ *iff* $f^k \xrightarrow{h} f$ *and* $f^k \xrightarrow{e} f$*. Hence* $f^k \xrightarrow{c} f$ *iff* $\operatorname{epi} f^k \to \operatorname{epi} f$ *and* $\operatorname{hyp} f^k \to \operatorname{hyp} f$*.*

4. *If* $f^k \xrightarrow{c} f$*, then* $f^{k_j}(x^{k_j}) \to f(x)$ *for any subsequence* $x^{k_j} \to x$*. Indeed, for such a subsequence, we define a sequence* $\{\tilde{x}^k\}$ *as follows:* $\tilde{x}^1 = x^{k_1}$, $\tilde{x}^2 = x^{k_1}$, $\ldots$, $\tilde{x}^{k_1-1} = x^{k_1}$, $\tilde{x}^{k_1} = x^{k_1}$, $\tilde{x}^{k_1+1} = x^{k_2}$, $\ldots$, $\tilde{x}^{k_2-1} = x^{k_2}$, $\tilde{x}^{k_2} = x^{k_2}$, $\tilde{x}^{k_2+1} = x^{k_3}$, $\ldots$*. Clearly,* $\tilde{x}^k \to x$ *and as* $f^k(\tilde{x}^k) \to f(x)$*, we have* $f^{k_j}(x^{k_j}) \to f(x)$*. Similarly, from* (9)–(10)*, we deduce that if* $f \leq c\text{-}\liminf_k f^k$ *(resp.* $c\text{-}\limsup_k f^k \leq f(x)$*), then* $f(x) \leq \liminf_j f^{k_j}(x^{k_j})$ *(resp.* $\limsup_j f^{k_j}(x^{k_j}) \leq f(x)$*) for any subsequence* $x^{k_j} \to x$*.*

5. [33] *A sequence* $\{f^k\}$ *is said to converge pointwise to* f*, denoted by* $f^k \xrightarrow{p} f$*, if* $f^k(x) \to f(x)$ *for all* x*; and to converge uniformly to* f *in the bounded sense, denoted by* $f^k \xrightarrow{u} f$*, if each* f^k *is bounded and for every* $\varepsilon > 0$ *there exists* N *such that* $|f^k(x) - f(x)| < \varepsilon$ *for all* x *and* $k \geq N$*. Clearly,* $f^k \xrightarrow{c} f$ *implies* $f^k \xrightarrow{p} f$*. If* $f^k \xrightarrow{u} f$ *with each* f^k *usc, then* f *is usc and* $f^k \xrightarrow{h} f$*.*

We recall the notion of continuous convergence to approximate multifunctions.

Definition 11. [33] *A sequence of multifunctions* $\{\Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n\}$ *is said to converge continuously to* Ψ*, denoted by* $\Psi^k \xrightarrow{c} \Psi$*, if* $\Psi^k(y^k) \to \Psi(y)$*, for every* $y^k \to y$*.*

As for the scalar case, we split continuous convergence into two parts.

Definition 12. *For a sequence of multifunctions* $\{\Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n\}$*, its continuous outer limit is the multifunction* $c\text{-}\limsup_k \Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ *defined by*

$$(c\text{-}\limsup_k \Psi^k)(y) := \bigcup_{\{y^k \to y\}} \limsup_k \Psi^k(y^k),$$

and its continuous inner limit is the multifunction $c\text{-}\liminf_k \Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ *defined by*

$$(c\text{-}\liminf_k \Psi^k)(y) := \bigcap_{\{y^k \to y\}} \liminf_k \Psi^k(y^k).$$

We have $c\text{-}\liminf_k \Psi^k \subset c\text{-}\limsup_k \Psi^k$*. When these limits are equal to* Ψ*, we say that the continuous limit* $c\text{-}\lim_k \Psi^k$ *exists and it is equal to* Ψ *and we write* $c\text{-}\lim_k \Psi^k = \Psi$*.*

Remark 13. 1. *Clearly,*

$$\Psi \subset c\text{-}\liminf_k \Psi^k \quad \Longleftrightarrow \quad \forall y \in \mathbb{R}^\ell, \forall y^k \to y\colon \Psi(y) \subset \liminf_k \Psi^k(y^k), \tag{11}$$

$$c\text{-}\limsup_k \Psi^k \subset \Psi \quad \Longleftrightarrow \quad \forall y \in \mathbb{R}^\ell, \forall y^k \to y\colon \limsup_k \Psi^k(y^k) \subset \Psi(y). \tag{12}$$

From this, it is easy to check that the following conditions are equivalent:

- $\Psi^k \xrightarrow{c} \Psi$*;*
- $c\text{-}\lim_k \Psi^k = \Psi$*;*
- $\Psi \subset c\text{-}\liminf_k \Psi^k$ *and* $c\text{-}\limsup_k \Psi^k \subset \Psi$*.*

Lignola and Morgan [22] *used the following terminology: A sequence* $\{\Psi^k\}$ *is sequentially lower (resp. upper) convergent to* Ψ*, if* $\Psi \subset c\text{-}\liminf_k \Psi^k$ *(resp.* $c\text{-}\limsup_k \Psi^k \subset \Psi$*).*

2. *If* $\Psi^k \xrightarrow{c} \Psi$*, then* Ψ *is continuous (see* [33, Theorem 5.43]*). In particular, if* $\Psi^k \equiv \Psi$*, then, to have* $\Psi^k \xrightarrow{c} \Psi$*, the map* Ψ *must be continuous. According to* [33]*, if* Ψ *is closed-valued, then* $\Psi^k \xrightarrow{c} \Psi$ *iff* $\lim_k \lim_{y \to \bar{y}} d_{\Psi^k(y)}(u) = d_{\Psi(\bar{y})}(u)$ *for all* $\bar{y}$ *and* u*.*

3. [33] *A sequence* $\{\Psi^k\}$ *is said to converge pointwise to* Ψ*, denoted by* $\Psi^k \xrightarrow{p} \Psi$*, if* $\Psi^k(y) \to \Psi(y)$ *for all* y*; and to converge uniformly to* Ψ*, denoted by* $\Psi^k \xrightarrow{u} \Psi$*, if for every* $\varepsilon > 0$ *and* $\rho > 0$*, there exists* N *such that* $\Psi^k(y) \cap \rho\mathbb{B} \subset \Psi(y) + \varepsilon\mathbb{B}$ *and* $\Psi(y) \cap \rho\mathbb{B} \subset \Psi^k(y) + \varepsilon\mathbb{B}$ *for all* y *and* $k \geq N$*. Clearly,* $\Psi^k \xrightarrow{c} \Psi$ *implies* $\Psi^k \xrightarrow{p} \Psi$*. By the metric version of uniform convergence in* [33, Proposition 5.49]*, we infer that* $\Psi^k \xrightarrow{u} \Psi$ *implies* $\Psi^k \xrightarrow{p} \Psi$*.*

Example 14. [22] 1. *Let* $\Psi^k(y) := [f^k(y), g^k(y)]$ *and* $\Psi(y) := [f(y), g(y)]$ *be interval functions where* $f^k, g^k, f, g \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *are functions with* $f^k \leq g^k$ *for all* k *and* $f \leq g$*. We have*

$$\begin{aligned} h\text{-}\limsup_k f^k \leq f \quad \text{and} \quad g \leq e\text{-}\liminf_k g^k \quad &\Longrightarrow \quad \Psi \subset c\text{-}\liminf_k \Psi^k \\ f \leq e\text{-}\liminf_k f^k \quad \text{and} \quad h\text{-}\limsup_k g^k \leq g \quad &\Longrightarrow \quad c\text{-}\limsup_k \Psi^k \subset \Psi. \end{aligned}$$

Hence, if $f^k \xrightarrow{c} f$ *and* $g^k \xrightarrow{c} g$*, then* $\Psi^k \xrightarrow{c} \Psi$*.*

2. *Let* $\Psi^k(y) \equiv C_k$ *for all* k *and* $\Psi(y) \equiv C$*, we have* $\Psi \subset c\text{-}\liminf_k \Psi^k$ *iff* $C \subset \liminf_k C_k$ *and* $c\text{-}\limsup_k \Psi^k \subset \Psi$ *iff* $\limsup_k C_k \subset C$*. Hence* $\Psi^k \xrightarrow{c} \Psi$ *iff* $C_k \to C$*.*

We recall the notion of graphical convergence to approximate multifunctions.

Definition 15. [33] *For a sequence of multifunctions* $\{\Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n\}$*, its graphical outer limit is the multifunction* $g\text{-}\limsup_k \Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ *whose graph is*

$$\operatorname{gph}(g\text{-}\limsup_k \Psi^k) = \limsup_k (\operatorname{gph} \Psi^k),$$

and its graphical inner limit is the multifunction $g\text{-}\liminf_k \Psi^k \colon \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ *whose graph is*

$$\operatorname{gph}(g\text{-}\liminf_k \Psi^k) = \liminf_k (\operatorname{gph} \Psi^k).$$

We have $g\text{-}\liminf_k \Psi^k \subset g\text{-}\limsup_k \Psi^k$ *and when these limits are equal to* Ψ*, we say that the graphical limit* $g\text{-}\lim_k \Psi^k$ *exists and that the sequence graph converges to* Ψ*, denoted by* $g\text{-}\lim_k \Psi^k = \Psi$ *or* $\Psi^k \xrightarrow{g} \Psi$*.*

Remark 16. 1. *Clearly,* $\Psi \subset g\text{-}\liminf_k \Psi^k$ *iff* $\operatorname{gph} \Psi \subset \liminf_k (\operatorname{gph} \Psi^k)$ *and* $g\text{-}\limsup_k \Psi^k \subset \Psi$ *iff* $\limsup_k (\operatorname{gph} \Psi^k) \subset \operatorname{gph} \Psi$*. Hence, the following conditions are equivalent:*

- $\Psi^k \xrightarrow{g} \Psi$*;*
- $\operatorname{gph} \Psi^k \to \operatorname{gph} \Phi$*;*
- $\Psi \subset g\text{-}\liminf_k \Psi^k$ *and* $g\text{-}\limsup_k \Psi^k \subset \Psi$*.*

We have the following formulas for graphical limits at every $y \in \mathbb{R}^\ell$ *(see* [33, Proposition 5.33]*):*

$$\begin{aligned}(g\text{-}\liminf_k \Psi^k)(y) &= \bigcup_{\{y_k \to y\}} \liminf_k \Psi^k(y^k),\\ (g\text{-}\limsup_k \Psi^k)(y) &= \bigcup_{\{y_k \to y\}} \limsup_k \Psi^k(y^k).\end{aligned}$$

Therefore,

$$\Psi \subset g\text{-}\liminf_k \Psi^k \iff \forall y \in \mathbb{R}^\ell\colon \Psi(y) \subset \bigcup_{\{y^k \to y\}} \liminf_k \Psi^k(y^k), \tag{13}$$

$$g\text{-}\limsup_k \Psi^k \subset \Psi \iff \forall y \in \mathbb{R}^\ell, \forall y^k \to y\colon \limsup_k \Psi^k(y^k) \subset \Psi(y). \tag{14}$$

2. *If* $\Psi^k \xrightarrow{g} \Psi$*, then* Ψ *is osc (see* [33, p. 167]*). In particular, if* $\Psi^k \equiv \Psi$*, then to have* $\Psi^k \xrightarrow{g} \Psi$*, the map* Ψ *must be osc.*

3. *By Definition 12 and part* (1)*, we have*

$$c\text{-}\liminf_k \Psi^k \subset g\text{-}\liminf_k \Psi^k \quad \textit{and} \quad c\text{-}\limsup_k \Psi^k = g\text{-}\limsup_k \Psi^k.$$

Hence

$$\begin{aligned}\Psi \subset c\text{-}\liminf_k \Psi^k &\implies \Psi \subset g\text{-}\liminf_k \Psi^k,\\ c\text{-}\limsup_k \Psi^k \subset \Psi &\iff g\text{-}\limsup_k \Psi^k \subset \Psi.\end{aligned}$$

From this, we infer that $\Psi^k \xrightarrow{c} \Psi$ *implies* $\Psi^k \xrightarrow{g} \Psi$*.*

4. [33] *In general, pointwise convergence does not imply graphical convergence and viceversa. A sequence can have different graphical and pointwise limits as shown in the next example. If* $\Psi^k \xrightarrow{u} \Psi$ *with each* Ψ^k *osc, then* Ψ *is osc and* $\Psi^k \xrightarrow{g} \Psi$*.*

5. [22] *A sequence* $\{\Psi^k\}$ *is said to open graph converge to* Ψ*, if for any* $(y, x) \in \operatorname{gph}\Psi$ *and any sequence* $(x^k, y^k) \to (x, y)$*, we have* $(y^k, x^k) \in \operatorname{gph}\Psi^k$ *for* k *large enough. Clearly, this notion implies* $\Psi \subset g\text{-}\liminf_k \Psi^k$*.*

Example 17. [33] *Let us consider the sequence* $\{\Psi^k\}$ *defined by*

$$\Psi^k(y) = \begin{cases} [0,1], & \text{if } 0 \le y \le 1 - 1/k;\\ [0, 2ky - 2k + 3], & \text{if } 1 - 1/k \le y \le 1 - 1/(2k);\\ [0, -2ky + 2k + 1], & \text{if } 1 - 1/(2k) \le y \le 1;\\ \emptyset, & \text{elsewhere.}\end{cases}$$

It converges graphically and pointwise to two different limits: $\Psi^k \xrightarrow{g} \widetilde{\Psi}$ *and* $\Psi^k \xrightarrow{p} \widehat{\Psi}$ *where*

$$\widetilde{\Psi}(y) = \begin{cases} [0,1], & \text{if } 0 \le y < 1;\\ [0,2], & \text{if } y = 1;\\ \emptyset, & \text{elsewhere.}\end{cases} \quad \textit{and} \quad \widehat{\Psi}(y) = \begin{cases} [0,1], & \text{if } 0 \le y \le 1;\\ \emptyset, & \text{elsewhere.}\end{cases}$$

We define nonempty-valuedness and boundedness notions for sequences of multifunctions. The second and the third notions appear in [23, 33].

Definition 18. *A sequence of multifunctions* $\{\Psi^k : \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n\}$ *is said to be*

- *eventually nonempty-valued (env), if* $\{\Psi^k(y)\}$ *is eventually nonempty-valued for every* y.
- *eventually uniformly bounded (eub), if* $\{\Psi^k(\mathbb{R}^m)\}$ *is eventually bounded.*
- *eventually locally bounded (elb), if* $\{\Psi^k(y^k)\}$ *is eventually bounded for every* $y^k \to y$.
- *strongly eventually locally bounded (selb), if* $\{\Psi^k(y^k)\}$ *is strongly eventually bounded for every* $y^k \to y$.

Remark 19. 1. *A sequence of nonempty-valued multifunctions is env. An eub sequence is elb. An elb sequence of nonempty-valued multifunctions is selb.*

2. *A sequence* $\{\Psi^k\}$ *is elb iff for every* y *there exist* $r > 0$, *a neighborhood* U *of* y, *and* N *such that* $\bigcup_{k\geq N} \Psi^k(U) \subset r\mathbb{B}$.

3. *When* $\Psi^k \equiv \Psi$, *then elb (resp. eub) property reduces to local (resp. uniform) boundedness of* Ψ.

4. *Lignola and Morgan [22] used the following equivalent formulation of elb property: "For any convergent sequence* $\{y^k\}$ *and any sequence* $\{x^k\}$ *such that* $x^k \in \Psi^k(y^k)$ *for all* k, *the sequence* $\{x^k\}$ *has a convergent subsequence".*

We prove some properties of these notions.

Proposition 20. (a) *If* $\{\Psi^k\}$ *is elb and* $\Psi \subset c\text{-}\liminf_k \Psi^k$, *then* Ψ *is locally bounded.*

(b) *If* $\{\Psi^k\}$ *is selb, then* $\limsup_k \Psi^k(y^k)$ *is nonempty and compact for every* $y^k \to y$.

(c) *If* $c\text{-}\limsup_k \Psi^k \subset \Psi$ *with* $\{\Psi^k\}$ *elb and* $\Psi(y) = \emptyset$ *for some* $y \in \mathbb{R}^m$, *then for every* $y^k \to y$ *there exists* $N \in \mathbb{N}$ *such that* $\Psi^k(y^k) = \emptyset$ *for all* $k \geq N$.

(d) *If* $\{\Psi^k\}$ *is eub and* $\Psi \subset g\text{-}\liminf_k \Psi^k$, *then* Ψ *is uniformly bounded.*

Proof. (a) For a fixed y there exist $r > 0$, a neighborhood U of y, and N such that $\Psi^k(z) \subset r\mathbb{B}$ for all $k \geq N$ and $z \in U$. By (11), we have $\Psi(z) \subset \liminf_k \Psi^k(z)$; thus, $\Psi(z) \subset r\mathbb{B}$ for all $z \in U$. Hence Ψ is locally bounded at y.

(b) The result follows from Proposition 1.

(c) Let $\Psi(y) = \emptyset$ and $y^k \to y$. By elb condition there exists $r > 0$ and N_1 such that $\bigcup_{k\geq N_1} \Psi^k(y^k) \subset r\mathbb{B}$. By (12), we have $\limsup_k \Psi^k(y^k) = \emptyset$. By the escaping to the horizon property, for such an r, there exists N_2 such that $\Psi^k(y^k) \cap r\mathbb{B} = \emptyset$ for all $k \geq N_2$, a contradiction if $\Psi^k(y^k) \neq \emptyset$ for some $k \geq N := \max\{N_1, N_2\}$.

(d) By hypothesis there exist $r > 0$ and N such that $\Psi^k(\mathbb{R}^m) \subset r\mathbb{B}$ for all $k \geq N$. Let y be fixed. If $x \in \Psi(y)$, then by (13) there exists $y^k \to y$ such that $x \subset \liminf_k \Psi^k(y^k)$. By the above inclusion, we have $x \in r\mathbb{B}$. Hence $\Psi(y) \subset r\mathbb{B}$. The result follows since y was arbitrary. □

We establish boundedness properties of the solution multifunctions of the approximations (4).

Proposition 21. *(a) If $\{\Phi^k\}$ is elb (resp. eub), then $\{S^k\}$ is elb (resp. eub).*

(b) If $\{\Phi^k\}$ is selb with Φ^k closed-valued and $u^k(\cdot, y)$ usc for all k and y, then $\{S^k\}$ is selb.

Proof. Part (a) follows from inclusion $S^k \subset \Phi^k$ for all k. We check part (b). Let $y^k \to y$. By (a) we have that $\{S^k\}$ is elb. By hypothesis there exist a subsequence $\{\Phi^{k_j}(y^{k_j})\}$ of nonempty sets. This and the hypothesis imply that the sets $\{S^{k_j}(y^{k_j})\}$ are nonempty. Hence $\{S^k\}$ is selb. □

3 Main results

In this section, we study the behavior of the value function and of the solution multifunction when the data of the parametric problem $(\mathcal{P}_y)$ are subject to variations. To vary the objective function and the feasible multifunction, we use variational convergence notions. We prove convergence properties of the sequence of value functions $\{v^k\}$ and solution multifunctions $\{\Phi^k\}$ defined in (4).

We first obtain a result that is a perturbed counterpart of Proposition 4(a).

Proposition 22. *Let $u \leq c\text{-}\liminf_k u^k$. Then*

(a) If $\Phi \subset g\text{-}\liminf_k \Phi^k$, Φ is compact-valued and $u(\cdot, y)$ is usc for all y, then $v \leq h\text{-}\liminf_k v^k$.

(b) If $\Phi \subset c\text{-}\liminf_k \Phi^k$, then $v \leq c\text{-}\liminf_k v^k$.

Proof. (a) If $y \notin \operatorname{dom}\Phi$, then $v(y) = -\infty$ and $\liminf_k v^k(y^k) \geq v(y)$ for all $y^k \to y$. On the other hand, if $y \in \operatorname{dom}\Phi$, then by hypothesis there exists $x \in \Phi(y)$ such that $v(y) = u(x, y)$. As $(y, x) \in \operatorname{gph}\Phi$, we have $(y, x) \in \liminf_k \operatorname{gph}\Phi^k$ and there exists $(y^k, x^k) \to (y, x)$ such that $x^k \in \Phi^k(y^k)$ for all k. As $v^k(y^k) \geq u^k(x^k, y^k)$ for all k, after taking the liminf, we obtain $\liminf_k v^k(y^k) \geq u(x, y)$. Hence $\liminf_k v^k(y^k) \geq v(y)$. The result follows by (5).

(b) Let $y^k \to y$. If $y \notin \operatorname{dom}\Phi$, then $v(y) = -\infty$ and $\liminf_k v^k(y^k) \geq v(y)$ holds for all $y^k \to y$. On the other hand, if $y \in \operatorname{dom}\Phi$, then by (11) for any $x \in \Phi(y)$ there exists $x^k \in \Phi^k(y^k) \to x$. As $v^k(y^k) \geq u^k(x^k, y^k)$ for all k, after taking the liminf, we obtain $\liminf_k v^k(y^k) \geq u(x, y)$. From this and since $x \in \Phi(y)$ was arbitrary, we obtain $\liminf_k v^k(y^k) \geq v(y)$. The result follows by (9). □

We now obtain the next result that is a perturbed counterpart of Proposition 4(b).

Proposition 23. *Let* c-$\limsup_k u^k \le u$ *and* c-$\limsup_k \Phi^k \subset \Phi$ *with* $\{\Phi^k\}$ *env and elb. Then*

(a) If Φ^k *is closed-valued and* $u^k(\cdot, y)$ *is usc for all* y *and* k*, then* c-$\limsup_k v^k \le v$*.*

(b) If $u(\cdot, y)$ *is continuous for all* y*, then* c-$\limsup_k v^k \le v$*.*

Proof. (a) Let $y^k \to y$. Consider the subsequence $\{v^{k_j}(y^{k_j})\}$ of $\{v^k(y^k)\}$ such that $\limsup_k v^k(y^k) = \lim_j v^{k_j}(y^{k_j})$. As $\{\Phi^{k_j}\}$ is elb and env, the sets $\Phi^{k_j}(y^{k_j})$ are nonempty and compact for j large enough. Since $u^{k_j}(\cdot, y^{k_j})$ is usc, there exists $x^{k_j} \in \Phi^{k_j}(y^{k_j})$ such that $v^{k_j}(y^{k_j}) = u^{k_j}(x^{k_j}, y^{k_j})$ for such j. By elb condition, there exists a subsequence $\{x^{k_{j_\ell}}\}$ of $\{x^{k_j}\}$ such that $x^{k_{j_\ell}} \to x$ for some x as $\ell \to +\infty$. As $x \in \limsup_k \Phi^k(y^k)$, by (12) we infer that $x \in \Phi(y)$. After taking the limit as $\ell \to +\infty$ to the last equality for subindex k_{j_ℓ} and by using Remark 10(4), we obtain

$$\limsup_k v^k(y^k) = \lim_\ell v^{k_{j_\ell}}(y^{k_{j_\ell}}) = \lim_\ell u^{k_{j_\ell}}(x^{k_{j_\ell}}, y^{k_{j_\ell}}) \le u(x, y) \le v(y).$$

Hence $\limsup_k v^k(y^k) \le v(y)$ and the result follows by (10).

(b) Let $y^k \to y$. Consider the subsequence $\{v^{k_j}(y^{k_j})\}$ of $\{v^k(y^k)\}$ such that $\limsup_k v^k(y^k) = \lim_j v^{k_j}(y^{k_j})$. We set $K := \limsup_j \Phi^{k_j}(y^{k_j})$. For an arbitrary $\varepsilon > 0$, by (12), we have

$$K \subset \limsup_k \Phi^k(y^k) \subset \Phi(y) \subset \{x \in \mathbb{R}^n \colon u(x, y) < v(y) + \varepsilon\}.$$

As K is nonempty and compact by Proposition 1 and $u(\cdot, y)$ is continuous, there exists $\delta > 0$ such that

$$K + \delta\mathbb{B} \subset \{x \in \mathbb{R}^n \colon u(x, y) < v(y) + \varepsilon\}.$$

By Proposition 1, there exists N such that

$$\bigcup_{k_j \ge N} \Phi^{k_j}(y^{k_j}) \subset K + \delta\mathbb{B} \subset \{x \in \mathbb{R}^n \colon u(x, y) < v(y) + \varepsilon\}.$$

Hence, for all $k_j \ge N$, we have

$$\begin{array}{rcl} v(y) + \varepsilon & \ge & \sup_{x \in K + \delta\mathbb{B}} u(x, y) \\ & \ge & \sup_{x \in \Phi^{k_j}(y^{k_j})} u(x, y) \\ & = & \sup_{x \in \Phi^{k_j}(y^{k_j})} [u^{k_j}(x, y^{k_j}) + (u(x, y) - u^{k_j}(x, y^{k_j}))] \\ & \ge & v^{k_j}(y^{k_j}) + \inf_{x \in \Phi^{k_j}(y^{k_j})} (u(x, y) - u^{k_j}(x, y^{k_j})). \end{array} \tag{15}$$

Let $\{x^{k_j}\}_{k_j \ge N}$ be a sequence such that $x^{k_j} \in \Phi^{k_j}(y^{k_j})$ and

$$u(x^{k_j}, y) - u^{k_j}(x^{k_j}, y^{k_j}) < \inf_{x \in \Phi^{k_j}(y^{k_j})} (u(x, y) - u^{k_j}(x, y^{k_j})) + \varepsilon, \ \forall k_j \ge N.$$

Since $K + \delta\mathbb{B}$ is compact and contains the sequence $\{x^{k_j}\}_{k_j \ge N}$, there exists a subsequence $\{x^{k_{j_\ell}}\}$ of $\{x^{k_j}\}_{k_j \ge N}$ such that $x^{k_{j_\ell}} \to x$ for some x as $\ell \to +\infty$. From this and (15) for index k_{j_ℓ}, we obtain

$$v(y) + u^{k_{j_\ell}}(x^{k_{j_\ell}}, y^{k_{j_\ell}}) > v^{k_{j_\ell}}(y^{k_{j_\ell}}) + u(x^{k_{j_\ell}}, y) - 2\varepsilon$$

and after taking the limsup as $l \to +\infty$ and by using Remark 10(4), we obtain

$$\begin{aligned} v(y) + u(x,y) &\geq v(y) + \limsup_\ell u^{k_{j_\ell}}(x^{k_{j_\ell}}, y^{k_{j_\ell}}) \\ &\geq \lim_\ell v^{k_{j_\ell}}(y^{k_{j_\ell}}) + u(x,y) - 2\varepsilon \\ &= \limsup_k v^k(y^k) + u(x,y) - 2\varepsilon. \end{aligned}$$

Hence $v(y) \geq \limsup_k v^k(y^k) - 2\varepsilon$. As $\varepsilon > 0$ was arbitrary, we obtain $\limsup_k v^k(y^k) \leq v(y)$ and the result follows by (10). □

We obtain stability properties of the value function and of the solution multifunction under both the continuous convergence of objective functions and feasible multifunctions.

Theorem 24. *Let $u^k \overset{c}{\to} u$ and $\Phi^k \overset{c}{\to} \Phi$. Then*

(a) *c-$\limsup_k S^k \subset S$ with S osc. In addition, if $\{\Phi^k\}$ is elb, then S is locally bounded; thus, S is compact-valued.*

(b) *If $\{\Phi^k\}$ is env and elb, then $v^k \overset{c}{\to} v$ with v continuous.*

(c) *If $\{\Phi^k\}$ is selb, Φ^k is closed-valued and $u^k(\cdot, y)$ is usc for all k and y, then $\{S^k\}$ is selb, c-$\limsup_k S^k$ is nonempty-valued, F_σ-valued, bounded-valued, and S is nonempty-valued.*

Proof. By Remarks 10(2) and 13(2), we infer that u and Φ are continuous.

(a) We prove the inclusion. Let $y^k \to y$. If $x \in \limsup_k S^k(y^k)$, then there exists $x^{k_j} \in S^{k_j}(y^{k_j}) \to x$. Hence $x^{k_j} \in \Phi^{k_j}(y^{k_j})$ and $v^{k_j}(y^{k_j}) = u^{k_j}(x^{k_j}, y^{k_j})$ for all j. As $x \in \limsup_k \Phi^k(y^k)$, by (12) we have $x \in \Phi(y)$. After taking the limits to the last equality, using Proposition 22(b), (10) and Remark 10(5), we obtain

$$v(y) \leq \liminf_k v^k(y^k) \leq \liminf_j v^{k_j}(y^{k_j}) = \lim_j u^{k_j}(x^{k_j}, y^{k_j}) = u(x,y);$$

i.e., $v(y) = u(x,y)$ and $x \in S(y)$. Hence $\limsup_k S^k(y^k) \subset S(y)$ and the inclusion follows from (12).

We have that S is osc by the first part since $\tilde{u}^k \overset{c}{\to} u$ and $\tilde{\Phi}^k \overset{c}{\to} \Phi$ hold for $\tilde{u}^k \equiv u$ and $\tilde{\Phi}^k \equiv \Phi$. The mapping S is locally bounded since $S \subset \Phi$ and Φ is locally bounded by Proposition 20(a). The last part follows from this and Proposition 3(a).

(b) The continuous convergence follows from Propositions 22(b) and 23(b). The continuity of v follows from Remark 10(2).

(c) The first part follows from Proposition 21. The remaining part follows from the first one, Proposition 1, Definition 12 and (a). □

From the proof of (a), we see that c-$\limsup_k u^k \leq u$ and c-$\limsup_k \Phi^k \subset \Phi$ imply c-$\limsup_k S^k \subset S$. A few remarks are needed concerning Theorem 24.

Remark 25. 1. *Assumption elb cannot be dropped. Indeed, for* $u^k(x,y) = \dfrac{1}{k}|x|$, $u(x,y) \equiv 0$ *and* $\Phi^k(y) = \Phi(y) \equiv \mathbb{R}$, *we have* $u^k \xrightarrow{c} u$, $\Phi^k \xrightarrow{c} \Phi$, $\{\Phi^k\}$ *is not elb,* $S^k(y) \equiv \emptyset$, $v^k(y) \equiv +\infty$, $v(y) \equiv 0$, *and* $S(y) \equiv \mathbb{R}$. *So (a)–(c) fail to hold. Clearly,* $S \not\subset c\text{-}\liminf_k S^k$ *and thus* $S^k \overset{c}{\not\to} S$.

2. *Continuous convergence of objective functions cannot be replaced by pointwise or uniform convergence. Indeed, for*

$$u^k(x,y) = u(x,y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{elsewhere.} \end{cases} \quad \text{and} \quad \Phi^k(y) = \Phi(y) \equiv [0,1],$$

we have $u^k \xrightarrow{p} u$, $u^k \xrightarrow{u} u$, $u^k \overset{c}{\not\to} u$, $\Phi^k \xrightarrow{c} \Phi$, $v^k(y) = v(y) \equiv 1$, $S^k(y) = S(y) \equiv [0,1] \cap \mathbb{Q}$ *and* $\lim_k S^k(y^k) = [0,1]$ *for every* $y^k \to y$. *So (a) fails to hold.*

3. *Continuous convergence of feasible multifunctions cannot be replaced by uniform convergence (and so by pointwise convergence). Indeed, for*

$$u^k(x,y) = u(x,y) \equiv 1 \quad \text{and} \quad \Phi^k(y) = \Phi(y) = \begin{cases} \{0\}, & \text{if } y \in \mathbb{Q}; \\ \{1\}, & \text{elsewhere,} \end{cases}$$

we have $u^k \xrightarrow{c} u$, $\Phi^k \xrightarrow{u} \Phi$, $\Phi^k \overset{c}{\not\to} \Phi$ *and*

$$S^k(y) = S(y) = \begin{cases} \{0\}, & \text{if } y \in \mathbb{Q}; \\ \{1\}, & \text{elsewhere.} \end{cases}$$

As $\limsup S^k(y^k) = \{0,1\}$ *for every* $y^k \to y$, *we have* $c\text{-}\limsup_k S^k \not\subset S$. *So (a) fails to hold.*

4. *The inclusion in (a) could be strict. Indeed, for* $u^k(x,y) = u(x,y) = f(x)$, $\Phi^k(y) = [0, 3+1/k]$, *and* $\Phi(y) \equiv [0,3]$ *where* $f \colon \mathbb{R} \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} x, & \text{if } x < 1; \\ 2 - x, & \text{if } 1 \le x < 2; \\ x - 2, & \text{if } 2 \le x, \end{cases}$$

we have $u^k \xrightarrow{c} u$, $\Phi^k \xrightarrow{c} \Phi$, $S^k(y) \equiv \{3 + 1/k\}$ *and* $S(y) \equiv \{1,3\}$. *Thus,* $\limsup_k S^k(y^k) \subsetneq S(y)$ *for every* $y^k \to y$.

5. *The concavity of objective functions and the convexity of feasible multifunctions (the graphs are convex) do not guarantee the equality in (a). Indeed, for* $u^k(x,y) = \dfrac{1}{k} f(x)$, $u(x,y) \equiv 0$, *and* $\Phi^k(y) = \Phi(y) \equiv [0,3]$ *where* $f : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} x, & \text{if } x < 1; \\ 1, & \text{if } 1 \le x < 2; \\ 3 - x, & \text{if } 2 \le x, \end{cases}$$

we have $u^k \xrightarrow{c} u$, $\Phi^k \xrightarrow{c} \Phi$, $S^k(y) \equiv [1,2]$ *and* $S(y) \equiv [0,3]$. *Thus,* $\limsup S^k(y^k) \subsetneq S(y)$ *for every* $y^k \to y$.

6. *It is important to point out that inclusion* $S \subset c\text{-}\liminf_k S^k$ *does not hold in general. Indeed, for* $u^k(x,y) = \dfrac{1}{k}|x|$, $u(x,y) \equiv 0$ *and* $\Phi^k(y) = \Phi(y) \equiv [-1,1]$, *we have* $u^k \xrightarrow{c} u$, $\Phi^k \xrightarrow{c} \Phi$, $v^k(y) \equiv 1/k$, $v(y) \equiv 0$, $S^k(y) \equiv \{-1,1\}$ *and* $S(y) \equiv [-1,1]$. *Thus,* $S(y) \not\subset \liminf_k S^k(y^k)$ *for every* $y^k \to y$.

As a consequence of these results, we infer an alternative version of Proposition 4 (see Remark 5).

Theorem 26. (*a*) *If* $u\colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ *is lsc and* $\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is isc, then* $v\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *is lsc.*

(*b*) *If* $u\colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ *is usc and* $\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is nonempty-valued, osc, locally bounded, then* $v\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *is usc.*

(*c*) *If* $u\colon \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ *is continuous,* $\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is nonempty-valued, continuous and locally bounded, then* $v\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ *is continuous and* $S\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is nonempty-valued, locally bounded and osc.*

Proof. Set $u^k \equiv u$ and $\Phi^k \equiv \Phi$ in Propositions 22(*b*) and 23(*a*) and Theorem 24. □

Remark 27. *It is worth pointing out that Theorem 26 holds for extended-valued functions and not only for finite functions as presented in the literature* (*cf.* [19, Proposition 3.4]). *Indeed, for* $u(x,y) = f(x)$ *and* $\Phi(y) = \{y\}$ *where* $f(x) = -\infty$ *if* $x \le -\pi/2$, $f(x) = \tan x$ *if* $x \in]-\pi/2, \pi/2[$ *and* $f(x) = +\infty$ *if* $x \ge \pi/2$, *the hypothesis of* (*c*) *holds. As* $v(y) = f(y)$ *and* $S(y) = \{y\}$, *the conclusions of* (*c*) *hold.*

We obtain stability properties of the value function and of the solution multifunction under continuous convergence of objective functions and graphical convergence of feasible multifunctions.

Theorem 28. *Let* $u^k \xrightarrow{c} u$ *and* $\Phi^k \xrightarrow{g} \Phi$. *Then*

(*a*) *If* $\{\Phi^k\}$ *is eub, then* S *is uniformly bounded; thus,* S *is compact-valued.*

(*b*) *If* $\{\Phi^k\}$ *is env and elb, then*

(*i*) $v^k \xrightarrow{h} v$ *with* v *usc.*

(*ii*) *For any* $y \in \mathbb{R}^m$ *there exists a sequence* $y^k \to y$ *such that* $\limsup_k S^k(y^k) \subset S(y)$.

Proof. By Remarks 10(2) and 16(2), we infer that u is continuous and Φ is osc.

(*a*) The mapping S is uniformly bounded since $S \subset \Phi$ and Φ is uniformly bounded by Proposition 20(*d*). The last part follows from this and the closed-valuedness of S since u is usc and Φ is closed-valued.

(*b*), (*i*) It follows from Propositions 22(*a*) and 23(*b*), and Remarks 10(4), 16(4) and 7(2).

(*ii*) Such a sequence $y^k \to y$ exists by (*i*) and Remark 7(1). If $x \in \limsup_k S^k(y^k)$, then there exists $x^{k_j} \in S^{k_j}(y^{k_j}) \to x$. As $x^{k_j} \in \Phi^{k_j}(y^{k_j}) \to x$, we have $x \in \limsup_k \Phi^k(y^k)$ that by (14) implies $x \in \Phi(y)$. After taking the liminf to $v^{k_j}(y^{k_j}) = u^{k_j}(x^{k_j}, y^{k_j})$, we obtain $v(y) \le u(x,y)$; i.e., $v(y) = u(x,y)$ and $x \in S(y)$. Hence $\limsup_k S^k(y^k) \subset S(y)$. □

Concerning Theorem 28, a few remarks are needed.

Remark 29. 1. *Continuous convergence of the function cannot be replaced by pointwise convergence or uniform convergence. Indeed, this is shown by Remark 25(2) where* $\Phi^k \xrightarrow{g} \Phi$.

2. *Let* $\Psi^k, \widetilde{\Phi}, \widehat{\Phi}$ *be the multifunctions in Example 17. For* $u^k(x,y) = u(x,y) = x$ *and* $\Phi^k = \Psi^k$, *we have* $u^k \xrightarrow{c} u$, $\{\Phi^k\}$ *is elb,*

$$v^k(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1 - \frac{1}{k}; \\ 2ky - 2k + 3, & \text{if } 1 - \frac{1}{k} \le y \le 1 - \frac{1}{2k}; \\ -2ky + 2k + 1, & \text{if } 1 - \frac{1}{2k} \le y \le 1; \\ +\infty, & \text{elsewhere;} \end{cases}$$

$$S^k(y) = \begin{cases} \{1\}, & \text{if } 0 \le y \le 1 - \frac{1}{k}; \\ \{2ky - 2k + 3\}, & \text{if } 1 - \frac{1}{k} \le y \le 1 - \frac{1}{2k}; \\ \{-2ky + 2k + 1\}, & \text{if } 1 - \frac{1}{2k} \le y \le 1; \\ \emptyset, & \text{elsewhere.} \end{cases}$$

The following convergence properties of $\{\Phi^k\}$, $\{v^k\}$ *and* $\{S^k\}$ *hold:*

- $\Phi^k \xrightarrow{g} \widetilde{\Phi}$ *where* $\widetilde{\Phi} = \widetilde{\Psi}$ *and* $v^k \xrightarrow{h} \widetilde{v}$ *where*

$$\widetilde{v}(y) = \begin{cases} 1, & \text{if } 0 \le y < 1; \\ 2, & \text{if } y = 1; \\ +\infty, & \text{elsewhere.} \end{cases}$$

Moreover, for such u *and* $\widehat{\Phi}$, *one has*

$$\widetilde{S}(y) = \begin{cases} \{1\}, & \text{if } 0 \le y < 1; \\ \{2\}, & \text{if } y = 1; \\ \emptyset, & \text{elsewhere.} \end{cases}$$

For $y = 1$ *and* $y^k = 1 - 1/k$, *we have* $\limsup_k S^k(y^k) = \{1\} \not\subset S(y) = \{2\}$. *Hence*

$$c\text{-}\limsup_k S^k \not\subset \widetilde{S} \ \ (\textit{or equivalently, } g\text{-}\limsup_k S^k \not\subset \widetilde{S}).$$

Contrary to the property expected that is for this instance (cf. Theorem 24).

- $\Phi^k \xrightarrow{p} \widehat{\Phi}$ *where* $\widehat{\Phi} = \widehat{\Psi}$. *For* u *and* $\widehat{\Phi}$, *we have*

$$\widehat{v}(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1; \\ +\infty, & \text{elsewhere,} \end{cases} \quad \text{and } \widehat{S}(y) = \begin{cases} \{1\}, & \text{if } 0 \le y \le 1; \\ \emptyset, & \text{elsewhere.} \end{cases}$$

For $y = 1$ *and* $y^k = 1 - 1/(2k)$, *we have* $\limsup_k v^k(y^k) \not\le \widehat{v}(y)$; *i.e.,* $v^k \overset{h}{\not\to} \widehat{v}$. *Hence graphical convergence of feasible multifunctions cannot be replaced by pointwise convergence. We point out that* $v^k \xrightarrow{h} \overline{v}$ *where*

$$\overline{v}(y) = \begin{cases} 1, & \text{if } 0 \le y < 1; \\ 2, & \text{if } y = 1; \\ +\infty, & \text{elsewhere.} \end{cases}$$

3. *Lignola and Morgan* [22, Propositions 4.3.1–4.3.2] *proved that* $v^k \xrightarrow{h} v$ *holds under assumptions that differ from ours. Namely, under each of the following assumptions:*

(i) c-$\limsup_k u^k \leq u$, *for any* (x,y) *there exists* $\tilde{x}^k \to x$ *such that* $u(x,y) \leq \liminf_k u^k(\tilde{x}^k, y^k)$ *for every* $y^k \to y$, *and* $\Phi^k \xrightarrow{c} \Phi$ *with* $\{\Phi^k\}$ *env and elb.*

(ii) $u^k \xrightarrow{h} u$, $\{\Phi^k\}$ *open graph converges to* Φ, *and* c-$\limsup_k \Phi^k \subset \Phi$ *with* $\{\Phi^k\}$ *env.*

Comparing with the hypothesis of Theorem 28(b). In (i), *assumption* $u \leq c$-$\liminf_k u^k$ *is weakened and the convergence notion for feasible multifunctions is strengthened. Whereas in* (ii), *the convergence notion for objective functions is weakened, condition elb is dropped, and assumption* $\Phi \subset g$-$\liminf_k \Phi^k$ *is strengthened.*

To obtain further stability properties of the solution multifunction, we recall the notion of ε-approximate solution for maximization problems defined by Loridan [24]. To do this, in what follows, we shall consider finite-valued objective functions and proper multifunction. For $\varepsilon > 0$ and $y \in \mathbb{R}^m$ being fixed, the set of ε-approximate solutions to problem $(\mathcal{P}_y)$ is defined by:

$$S^{\varepsilon}(y) := \{x \in \Phi(y) \colon v(y) - \varepsilon \leq u(x,y)\}.$$

We list some properties of this notion:

- If $y \notin \operatorname{dom}\Phi$, then $v(y) = -\infty$ and $S^{\varepsilon}(y) = \emptyset$.
- If $y \in \operatorname{dom}\Phi$, then $-\infty < v(y) \leq +\infty$. In this case, if $v(y) = +\infty$, then $S^{\varepsilon}(y) = \emptyset$. Hence $v(y) < +\infty$ iff $S^{\varepsilon}(y) \neq \emptyset$.
- If $0 < \varepsilon < \varepsilon'$, then $S(y) \subset S^{\varepsilon}(y) \subset S^{\varepsilon'}(y)$ and $S(y) = \bigcap_{\varepsilon > 0} S^{\varepsilon}(y)$.
- If $u(\cdot, y)$ is usc and $\Phi(y)$ is closed, then $S^{\varepsilon}(y)$ is closed and $\lim_k S^{\varepsilon_k}(y) = \bigcap_k \operatorname{cl} S^{\varepsilon_k}(y)$ for every $\varepsilon_k \downarrow 0$. Hence, if $u^k(\cdot, y)$ is usc and $\Phi^k(y)$ is closed for all y and k, then $\lim_k S^{\varepsilon_k}(y) = S(y)$ for all y; i.e., $S^{\varepsilon_k} \xrightarrow{p} S$.

For approximations in (4), we define

$$S^{k,\varepsilon}(y) := \{x \in \Phi^k(y) \colon v^k(y) - \varepsilon \leq u^k(x,y)\}.$$

We establish the 'continuous convergence nesting' for problem $(\mathcal{P}_y)$ that is a counterpart of the 'epigraphical nesting' in [33, Proposition 7.30] for minimization problems.

Proposition 30. *Let* $u \leq c$-$\liminf_k u^k$ *and* $\Phi \subset c$-$\liminf_k \Phi^k$. *Then*

$$v \leq c\text{-}\liminf_k v^k.$$

In addition, if $c\text{-}\limsup_k \Phi^k \subset \Phi$*, then*

$$c\text{-}\limsup_k S^{k,\varepsilon_k} \subset S$$

for any $\varepsilon_k \searrow 0$ *such that whenever* $(y^{k_j}, x^{k_j}) \in \operatorname{gph} S^{k_j,\varepsilon_{k_j}} \to (y,x)$*, then* $u^{k_j}(x^{k_j}, y^{k_j}) \to u(x,y)$*.*

Proof. The first part appears in Proposition 22(b). We check the second one. Let y be fixed and $\varepsilon_k \searrow 0$ be that from the formulation. If $x \in (c\text{-}\limsup_k S^{k,\varepsilon_k})(y)$, then by Definition 12 there exists $y^k \to y$ such that $x \in \limsup_k S^{k,\varepsilon_k}(y^k)$. Hence there exists $x^{k_j} \in S^{k_j,\varepsilon_{k_j}}(y^{k_j}) \to x$. Therefore

$$x^{k_j} \in \Phi^{k_j}(y^{k_j}) \text{ and } v^{k_j}(y^{k_j}) - \varepsilon_{k_j} \leq u^{k_j}(x^{k_j}, y^{k_j}), \forall j \in \mathbb{N}. \tag{16}$$

As $x \in \limsup_k \Phi^k(y^k)$, by (12) we have $x \in \Phi(y)$. By Proposition 22(b), we have $v \leq c\text{-}\liminf_k v^k$; thus, $v(y) \leq \liminf_k v^k(y^k)$ by (9). By hypothesis, we have $u^{k_j}(x^{k_j}, y^{k_j}) \to u(x,y)$ as $j \to +\infty$. After taking the limit in (16) as $j \to +\infty$, we obtain $v(y) \leq u(x,y)$; i.e., $v(y) = u(x,y)$. Hence $x \in S(y)$ and the inclusion follows. □

We extend the 'argmax' part of [33, Proposition 7.31] to problem $(\mathcal{P}_y)$.

Proposition 31. *Let* $u^k \xrightarrow{c} u$ *and* $\Phi^k \xrightarrow{c} \Phi$*. Then*

(a) $c\text{-}\limsup_k S^{k,\varepsilon} \subset S^\varepsilon$ *for every* $\varepsilon > 0$ *and* $c\text{-}\limsup_k S^{k,\varepsilon_k} \subset S$ *for every* $\varepsilon_k \searrow 0$*.*

(b) *If* $\{\Phi^k\}$ *is env and elb, then* $S^\varepsilon \subset c\text{-}\liminf_k S^{k,2\varepsilon}$ *for every* $\varepsilon > 0$*. Hence*

$$S = \bigcap_{\varepsilon>0} c\text{-}\liminf_k S^{k,\varepsilon} = \bigcap_{\varepsilon>0} c\text{-}\limsup_k S^{k,\varepsilon}.$$

In addition, if u^k *is usc and* Φ^k *is closed-valued for all* k*, then*

$$c\text{-}\lim_k S^{k,\varepsilon_j} \xrightarrow{p} S \text{ as } j \to +\infty \text{ for every } \varepsilon_j \downarrow 0.$$

Proof. The proof of (a) runs as in Proposition 30.

(b) We prove the inclusion. Let $y^k \to y$. If $S^\varepsilon(y) = \emptyset$, then $S^\varepsilon(y) \subset \liminf_k S^{k,2\varepsilon}(y^k)$. On the other hand, if $S^\varepsilon(y) \neq \emptyset$, then for $x \in S^\varepsilon(y)$, we have $x \in \Phi(y)$ and $v(y) - \varepsilon \leq u(x,y)$ with $v(y)$ finite. As $\Phi(y) \subset \liminf_k \Phi^k(y^k)$, by (11) there exists $x^k \in \Phi^k(y^k) \to x$. Since $v^k(y^k) \to v(y)$ by Theorem 24, $u^k(x^k, y^k) \to u(x,y)$, and $v(y) - 2\varepsilon < u(x,y)$, we infer that $v^k(y^k) - 2\varepsilon < u^k(x^k, y^k)$ for k large enough. Hence $x^k \in S^{k,2\varepsilon}(y^k) \to x$; i.e., $x \in \liminf_k S^{k,2\varepsilon}(y^k)$. Thus, $S^\varepsilon(y) \subset \liminf_k S^{k,2\varepsilon}(y^k)$ and the result follows from (11).

We check the equalities. The third inclusion and (a) imply

$$S^{\varepsilon/2} \subset c\text{-}\liminf_k S^{k,\varepsilon} \subset c\text{-}\limsup_k S^{k,\varepsilon} \subset S^\varepsilon \tag{17}$$

The equality follows after taking the intersection w.r.t. $\varepsilon > 0$. The pointwise limit follows by setting ε_j in (17) and since $S^{\varepsilon_j/2} \xrightarrow{p} S$ and $S^{\varepsilon_j} \xrightarrow{p} S$ as $j \to +\infty$. □

Proposition 32. *Let $u^k \xrightarrow{c} u$ and $\Phi^k \xrightarrow{g} \Phi$ with $\{\Phi^k\}$ env and elb. Then*

(*a*) *For any $y \in \mathbb{R}^m$ there exists a sequence $y^k \to y$ such that $v^k(y^k) \to v(y)$, $\limsup_k S^{k,\varepsilon}(y^k) \subset S^\varepsilon(y)$ for every $\varepsilon > 0$ and $\limsup_k S^{k,\varepsilon_k}(y^k) \subset S(y)$ for every $\varepsilon_k \searrow 0$.*

(*b*) *$S^\varepsilon \subset g\text{-}\liminf_k S^{k,2\varepsilon}$ for every $\varepsilon > 0$ and $S \subset \bigcap_{\varepsilon>0} g\text{-}\liminf_k S^{k,\varepsilon}$.*

Proof. (*a*) Such a sequence $y^k \to y$ exists by Theorem 28(*b*) and Remark 7(1). If $x \in \limsup_k S^{k,\varepsilon}(y^k)$, then there exists $x^{k_j} \in S^{k_j,\varepsilon}(y^{k_j}) \to x$. Hence $x^{k_j} \in \Phi^{k_j}(y^{k_j})$ and $v^{k_j}(y^{k_j}) - \varepsilon \le u^{k_j}(x^{k_j}, y^{k_j})$ for all j. Thus, $(y,x) \in \limsup_k \operatorname{gph} \Phi^k$ that by (14) implies $(y,x) \in \operatorname{gph} \Phi$; i.e., $x \in \Phi(y)$. After taking the limit to the above inequality, we obtain $v(y) - \varepsilon \le u(x,y)$; i.e., $x \in S^\varepsilon(y)$ and the first inclusion follows. The second one follows similarly.

(*b*) Let $y \in \mathbb{R}^m$. If $x \in S^\varepsilon(y)$, then $x \in \Phi(y)$ and $v(y) - \varepsilon \le u(x,y)$. By (13) there exists $y^k \to y$ such that $x \in \liminf_k \Phi^k(y^k)$. Hence there exists $x^k \in \Phi^k(y^k) \to x$. By Theorem 28(*b*) and (6), we have $\limsup_k v^k(y^k) \le v(y)$. Clearly, there exists $N_1 \in \mathbb{N}$ such that $v^k(y^k) < v(y) + \varepsilon/2$ for all $k \ge N_1$. Similarly, as $u^k(x^k, y^k) \to u(x,y)$ there exists $N_2 \in \mathbb{N}$ such that $v(y) - 3\varepsilon/2 < u^k(x^k, y^k)$ for all $k \ge N_2$. Therefore, $v^k(y^k) - 2\varepsilon < u^k(x^k, y^k)$ for all $k \ge \max\{N_1, N_2\}$; i.e., $x^k \in S^{k,2\varepsilon}(y^k) \to x$; thus, $x \in \liminf_k S^{k,2\varepsilon}(y^k)$. Hence $S^\varepsilon(y) \subset \bigcup_{\{y^k \to y\}} \liminf_k S^{k,2\varepsilon}(y^k)$ and the result follows from (13). The remaining follows by taking the intersection w.r.t. $\varepsilon > 0$. □

Finally, we establish stability properties of the value function $v(y)$ and of the solution multifunction $S(y)$ at a given fixed point $y \in \mathbb{R}^m$. They follow straightforwardly from the convergence properties for minimization problems in terms epi-convergence in [33, 34]. Their maximization counterparts are obtained from the relationship between epi- and hypo-convergence in Remark 7(3).

To obtain these properties, we express the value function as a supremum of a new objective function over the whole space $\mathbb{R}^n$ by using indicator functions as follows:

$$v(y) = \sup_{x \in \mathbb{R}^n} \{u(x,y) - \delta_{\Phi(y)}(x)\} \quad \text{and} \quad v^k(y) = \sup_{x \in \mathbb{R}^n} \{u^k(x,y) - \delta_{\Phi^k(y)}(x)\}.$$

We denote the new objective functions by $f_y(x) := u(x,y) - \delta_{\Phi(y)}(x)$ and $f_y^k(x) := u^k(x,y) - \delta_{\Phi^k(y)}(x)$.

We establish convergence assumptions on the data $\{u^k\}$ and $\{\Phi^k\}$ that imply the hypo-convergence of the sequence $\{f_y^k\}$. To this end, we use the following properties for sets $\{C_k\}$ and C from $\mathbb{R}^\ell$:

$$C = \liminf_k C_k \iff \delta_C = e\text{-}\limsup_k \delta_{C_k} \quad \text{and} \quad C = \limsup_k C_k \iff \delta_C = e\text{-}\liminf_k \delta_{C_k}. \qquad (18)$$

These equivalences follow from equality $\operatorname{epi} \delta_C = C \times \mathbb{R}_+$ and properties of the limits of Cartesian products (see [33, Proposition 7.4]).

Proposition 33. (*a*) *If $u(\cdot,y) \le c\text{-}\liminf_k u^k(\cdot,y)$ and $\Phi(y) \subset \liminf_k \Phi^k(y)$, then $f_y \le h\text{-}\liminf_k f_y^k$.*

(*b*) *If $c\text{-}\limsup_k u^k(\cdot,y) \le u(\cdot,y)$ and $\limsup_k \Phi^k(y) \subset \Phi(y)$, then $h\text{-}\limsup_k f_y^k \le f_y$.*

(*c*) *If* $u^k(\cdot, y) \xrightarrow{c} u(\cdot, y)$ *and* $\Phi^k(y) \to \Phi(y)$*, then* $f_y^k \xrightarrow{h} f_y$*.*

Proof. We use (18), Remark 7(3) to write epi-limits as hypo-limits, and $A \subset B$ iff $\delta_B \leq \delta_A$.

(*a*) As $e\text{-}\limsup_k \delta_{\Phi^k(y)} \leq \delta_{\Phi(y)}$, we have $-\delta_{\Phi(y)} \leq h\text{-}\liminf_k(-\delta_{\Phi^k(y)})$. By (5), for every $x \in \mathbb{R}^n$ there exists $x^k \to x$ such that $\liminf_k(-\delta_{\Phi^k(y)}(x^k)) \geq -\delta_{\Phi(y)}(x)$; thus,

$$\liminf_k f_y^k(x^k) \geq \liminf_k u^k(x^k, y) + \liminf_k(-\delta_{\Phi^k(y)}(x^k)) \geq u(x, y) - \delta_{\Phi(y)}(x) = f_y(x),$$

and $f_y \leq h\text{-}\liminf_k f_y^k$ by (5).

(*b*) As $\delta_{\Phi(y)} \leq e\text{-}\liminf_k \delta_{\Phi^k(y)}$, we have $h\text{-}\limsup(-\delta_{\Phi^k(y)}) \leq -\delta_{\Phi(y)}$. By (6), for every $x^k \to x$, we have $\limsup_k(-\delta_{\Phi^k(y)}(x^k)) \leq -\delta_{\Phi(y)}(x)$. Hence

$$\limsup_k f_y^k(x^k) \leq \limsup_k u^k(x^k, y) + \limsup_k(-\delta_{\Phi^k(y)}(x^k)) \leq u(x, y) - \delta_{\Phi(y)}(x) = f_y(x),$$

and $h\text{-}\limsup_k f_y^k \leq f_y$ by (6).

(*c*) It follows from (*a*)–(*b*). □

Remark 34. *Zolezzi* [37] *assumed variational convergence or epi-convergence of* $\{f_y^k\}$ *to* f_y *to study the stability of problem* $(\mathcal{P}_y)$ *for a fixed* $y \in \mathbb{R}^m$*. He did not establish convergence assumptions on* $\{u^k\}$ *and* $\{\Phi^k\}$ *that ensure such a convergence.*

The next result is a consequence of the convergence in minimization in [33, 34] and Remark 7(3).

Theorem 35. (*a*) *If* $u(\cdot, y) \leq c\text{-}\liminf_k u^k(\cdot, y)$ *and* $\Phi(y) \subset \liminf_k \Phi^k(y)$*, then* $v(y) \leq \liminf_k v^k(y)$*.*

(*b*) *If* $c\text{-}\limsup_k u^k(\cdot, y) \leq u(\cdot, y)$ *with* $u^k(\cdot, y)$ *usc for all* k *and* $\limsup_k \Phi^k(y) \subset \Phi(y)$ *with* $\{\Phi^k(y)\}$ *eventually bounded with closed sets, then* $\limsup_k v^k(y) \leq v(y)$*.*

(*c*) *If* $u^k(\cdot, y) \xrightarrow{c} u(\cdot, y)$ *with* $u^k(\cdot, y)$ *usc for all* k *and* $\Phi^k(y) \to \Phi(y)$ *with* $\{\Phi^k(y)\}$ *eventually bounded with closed sets, then* $v^k(y) \to v(y)$*.*

(*d*) *If* $u^k(\cdot, y) \xrightarrow{c} u(\cdot, y)$ *with* $u(\cdot, y)$ *being* U*-proper and* $\Phi^k(y) \to \Phi(y)$*, then for any* $\varepsilon \in [0, +\infty)$*:*

(*i*) $\limsup_k S^{k,\varepsilon_k}(y) \subset S^\varepsilon(y)$ *for every* $\varepsilon_k \in [0, +\infty[\to \varepsilon$*. In particular,* $\limsup_k S^k(y) \subset S(y)$*.*

(*ii*) *If for some subsequence* $\varepsilon_{k_j} \in [0, +\infty[\to 0$ *there exists a convergent subsequence* $x^{k_j} \in S^{k_j,\varepsilon_{k_j}}(y)$ *for all* j*, then* $v^{k_j}(y) \to v(y)$*.*

(*iii*) *If* $v^k(y) \to v(y)$*, then there exists* $\varepsilon_k \in [0, +\infty[\to \varepsilon$ *such that* $S^\varepsilon(y) \subset \liminf_k S^{k,\varepsilon_k}(y)$*.*

Proof. (*a*) By Proposition 33(*a*), we have $f_y \leq h\text{-}\liminf_k f_y^k$. This, Remark 7(3) and [33, Proposition 7.30] imply $\liminf_k(\sup_{\mathbb{R}^n} f_y^k) \geq \sup_{\mathbb{R}^n} f_y$.

(*b*) Clearly, $\limsup_k v^k(y) = \lim_j v^{k_j}(y)$ for some subsequence $\{v^{k_j}(y)\}$ of $\{v^k(y)\}$. If there exists N such that $\Phi^{k_j}(y) = \emptyset$ for all $j \geq N$, then $v^{k_j}(y) = -\infty$ and $\limsup_k v^k(y) \leq v(y)$ holds. On the contrary, if for each $\ell \in \mathbb{N}$ there exists $k_{j_\ell} \geq \ell$ such that $\Phi^{k_{j_\ell}}(y) \neq \emptyset$. By hypothesis there exists $x^{k_{j_\ell}} \in \Phi^{k_{j_\ell}}(y)$ such that $v^{k_{j_\ell}}(y) = u^{k_{j_\ell}}(x^{k_{j_\ell}}, y)$ for ℓ large enough. As $\{x^{k_{j_\ell}}\}$ is bounded since elb, we have $x^{k_{j_\ell}} \to x$ for some x, up to subsequences. Hence $x \in \limsup_k \Phi^k(y)$ and thus $x \in \Phi(y)$. By taking the limit to the last equality, we have $\limsup_k v^k(y) = \lim_\ell v^{k_{j_\ell}}(y) = \lim_\ell u^{k_{j_\ell}}(x^{k_{j_\ell}}, y) \leq u(x, y) \leq v(y)$.

(c) It follows from (a)–(b).

(d) By Proposition 33(c), we have $f_y^k \xrightarrow{h} f_y$. As f_y is U-proper, by Remark 7(3) and [34, Theorem 7.5] we infer that (i)–(iii) hold. □

4 Applications

We apply our results to the study the stability of the generalized Nash equilibrium problem and of the finite-horizon dynamic programming model.

4.1 Stability of generalized Nash equilibrium problems

Contrary to optimization problems, there is not a significant literature on the study of approximation of generalized Nash equilibria. For instance, Morgan and Raucci [29], and Gürkan and Pang [17] basically focused on the approximation of Nash equilibria. In this section, we address the issue of the convergence of generalized Nash equilibria.

Let N be the set of players, which is a nonempty and finite set. Let us assume that each player, labeled by $\nu \in N$, chooses a strategy x^ν in a strategy set K_ν, which is a subset of $\mathbb{R}^{n_\nu}$. We define the Cartesian products $\mathbb{R}^n := \prod_{\nu \in N} \mathbb{R}^{n_\nu}$ where $n = \sum_{\nu \in N} n_\nu$, $K := \prod_{\nu \in N} K_\nu$, and $K_{-\nu} := \prod_{\mu \in N \setminus \{\nu\}} K_\mu$ for $\nu \in N$. We write $x = (x_\nu, x_{-\nu}) \in K$ in order to emphasize the strategy of player ν, $x_\nu \in K_\nu$, and the strategy of the other players $x_{-\nu} \in K_{-\nu}$. Given the strategy of the players except of player ν, $x_{-\nu}$, the player ν chooses a strategy x_ν, solving the following optimization problem:

$$\max_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}), \quad \text{subject to} \quad x_\nu \in K_\nu, \tag{19}$$

where $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function, and $\theta_\nu(x_\nu, x_{-\nu})$ denotes the loss suffered by the player ν, when the rival players have chosen the strategy $x_{-\nu}$. Thus, a *Nash equilibrium* [31] is a vector $\hat{x} \in K$, such that $\hat{x}_\nu$ solves (19), when the rival players take the strategy $\hat{x}_{-\nu}$, for any $\nu \in N$. We denote by $\mathrm{NEP}(\{\theta_\nu, K_\nu\}_{\nu \in N})$ the set of Nash equilibria.

The necessity of a generalization of the Nash equilibrium problem arose, involving player interactions at the feasible sets level. Arrow and Debreu [2] termed it as *abstract economy* but nowadays, it is called the generalized Nash equilibrium problem. Recently, it gained more and more attention because it models real problems as electricity markets, environmental games, bilateral exchanges of bads, among others, see for instance [6, 14, 20, 36].

Formally, in a generalized Nash equilibrium problem [15], each player's strategy must belong to a set $X_\nu(x_{-\nu}) \subset K_\nu$ depending on the rival players' strategies. The aim of player ν, given the others players' strategies $x_{-\nu}$, is to choose a strategy x_ν that solves the next maximization problem

$$\max_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}), \quad \text{subject to} \;\; x_\nu \in X_\nu(x_{-\nu}), \tag{20}$$

where X_ν is a multifunction from $\mathbb{R}^{n-n_{-\nu}}$ to $\mathbb{R}^{n_\nu}$ such that $X_\nu(\mathbb{R}^{n-n_\nu}) \subset K_\nu$. Thus, a vector $\hat{x} \in \mathbb{R}^n$ is a *generalized Nash equilibrium* if, $\hat{x}_\nu$ solves (20) when its rival players take the strategy $\hat{x}_{-\nu}$, for any $\nu \in N$. It is clear that if $\hat{x}$ is a generalized Nash equilibrium, then $\hat{x} \in K$. We denote by $\text{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N})$ the set of generalized Nash equilibria.

For each $\nu \in N$, we consider the best response multifunction $S_\nu : \mathbb{R}^{n-n_\nu} \rightrightarrows K_\nu$, defined by

$$S_\nu(x_{-\nu}) := \arg\max_{X_\nu(x_{-\nu})} \theta_\nu(\cdot, x_{-\nu}),$$

and the function $v_\nu : \mathbb{R}^{n-n_\nu} \to \overline{\mathbb{R}}$, defined by

$$v_\nu(x_{-\nu}) := \sup_{x_\nu \in X_\nu(x_{-\nu})} \theta_\nu(x_\nu, x_{-\nu}).$$

It is not difficult to verify that

$$\text{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N}) = \bigcap\nolimits_{\nu\in N} \text{gph} S_\nu.$$

Next, we study convergence properties of approximations $\{\theta_\nu^k\}_{\nu\in N}$ and $\{X_\nu^k\}_{\nu\in N}$ of the objective function and of the feasible multifunction, respectively.

Theorem 36. *Let $\theta_\nu^k \xrightarrow{c} \theta_\nu$ and $X_\nu^k \xrightarrow{c} X_\nu$, for every $\nu \in N$. Then*

(*a*) $\limsup_k \text{GNEP}(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N}) \subset \text{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N})$.

(*b*) *For each $\nu \in N$, if $\{X_\nu^k\}$ is elb and env, then $v_\nu^k \xrightarrow{c} v_\nu$.*

Proof. (*a*) By Theorem 24(*a*), we deduce that $\limsup_k (\text{gph} S_\nu^k) \subset \text{gph} S_\nu$ for each $\nu \in N$. Thus,

$$\begin{aligned}
\limsup_k \text{GNEP}(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N}) &= \limsup_k \bigcap\nolimits_{\nu\in N} \text{gph} S_\nu^k \\
&\subset \bigcap\nolimits_{\nu\in N} \limsup_k (\text{gph} S_\nu^k) \\
&\subset \bigcap\nolimits_{\nu\in N} \text{gph} S_\nu \\
&= \text{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N}).
\end{aligned}$$

(*b*) It follows from Theorem 24(*b*). □

Remark 37. 1. *Part* (*a*) *of Theorem 36 is an extension of* [13] *and* [17, Theorem 1] *to generalized Nash games, where instead of multi-epiconvergence used in* [17] *we consider continuous convergence.*

2. *A natural question that arises is whether it is possible to obtain that:*

$$\mathrm{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N}) \subset \liminf_k \mathrm{GNEP}(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N})?$$

The answer to this question is negative, as shown in [29, Example 1.1] *for Nash equilibria.*

For each $\varepsilon > 0$, a vector $\hat{x} \in \mathbb{R}^n$ is said to be an *ε-approximate generalized Nash equilibrium*, if

$$\hat{x}_\nu \in X_\nu(\hat{x}_{-\nu}) \text{ and } v_\nu(\hat{x}_{-\nu}) - \varepsilon \leq \theta_\nu(x_\nu, \hat{x}_{-\nu}), \text{ for all } \nu \in N.$$

We denote by $\mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}_{\nu\in N})$ the set of ε-approximate generalized Nash equilibria. Clearly

$$\mathrm{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N}) \subset \mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}_{\nu\in N}),\ \forall \varepsilon > 0.$$

For each $\nu \in N$, we define the multifunction $S_{\nu,\varepsilon} : \mathbb{R}^{n-n_\nu} \rightrightarrows \mathbb{R}^{n_\nu}$ by

$$S_{\nu,\varepsilon}(x_{-\nu}) := \{x_\nu \in X_\nu(x_{-\nu}) : \ v_\nu(x_{-\nu}) - \varepsilon \leq \theta_\nu(x_\nu, x_{-\nu})\}.$$

Clearly, $\mathrm{gph} S_\nu \subset \mathrm{gph} S_{\nu,\varepsilon}$ and $\mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}_{\nu\in N}) = \bigcap_{\nu\in N} \mathrm{gph} S_{\nu,\varepsilon}$.

The following example illustrates the previous definition.

Example 38. Let $K_1 = K_2 = [0, +\infty[$, X_1, X_2 be constant multifunctions that are equal to $[0, +\infty[$, and $\theta_1, \theta_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be functions defined by $\theta_1(x, y) = 1 - e^{-x}$ and $\theta_2(x, y) = 1$. Clearly,

$$\mathrm{GNEP}(\{\theta_\nu, X_\nu\}) = \emptyset \quad \text{and} \quad \mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}) = [-\ln\varepsilon, +\infty[\times[0, +\infty[.$$

The following result extends [29, Proposition 1.13] to generalized Nash games.

Theorem 39. *Let $\theta_\nu^k \xrightarrow{c} \theta_\nu$ and $X_\nu^k \xrightarrow{c} X_\nu$, for every $\nu \in N$. Then*

(*a*) $\limsup_k \mathrm{GNEP}_\varepsilon(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N}) \subset \mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}_{\nu\in N})$*, for all* $\varepsilon > 0$.

(*b*) $\limsup_k \mathrm{GNEP}_{\varepsilon_k}(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N}) \subset \mathrm{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N})$*, for all* $\varepsilon_k \searrow 0$.

Proof. (*a*) By Theorem 31(*a*), we have $\limsup_k(\mathrm{gph} S_{\nu,\varepsilon}^k) \subset \mathrm{gph} S_{\nu,\varepsilon}$ for each $\nu \in N$. Thus,

$$\begin{aligned}\limsup_k \mathrm{GNEP}_\varepsilon(\{\theta_\nu^k, X_\nu^k\}_{\nu\in N}) &= \limsup_k \bigcap_{\nu\in N} \mathrm{gph} S_{\nu,\varepsilon}^k \\ &\subset \bigcap_{\nu\in N} \limsup_k(\mathrm{gph} S_{\nu,\varepsilon}^k) \\ &\subset \bigcap_{\nu\in N} \mathrm{gph} S_{\nu,\varepsilon} \\ &= \mathrm{GNEP}_\varepsilon(\{\theta_\nu, X_\nu\}_{\nu\in N}).\end{aligned}$$

(b) Once again, by Theorem 31(a), we have $\limsup_k(\mathrm{gph}S^k_{\nu,\varepsilon_k}) \subset \mathrm{gph}S_\nu$ for each $\nu \in N$. Thus,

$$\begin{aligned}
\limsup_k \mathrm{GNEP}_{\varepsilon_k}(\{\theta^k_\nu, X^k_\nu\}_{\nu\in N}) &= \limsup_k \bigcap\nolimits_{\nu\in N} \mathrm{gph}S^k_{\nu,\varepsilon_k} \\
&\subset \bigcap\nolimits_{\nu\in N} \limsup_k(\mathrm{gph}S^k_{\nu,\varepsilon_k}) \\
&\subset \bigcap\nolimits_{\nu\in N} \mathrm{gph}S_\nu \\
&= \mathrm{GNEP}(\{\theta_\nu, X_\nu\}_{\nu\in N}).
\end{aligned}$$

□

Remark 40. Theorem 39 with $X^k_\nu \equiv K_\nu$ for all $\nu \in N$, can be deduced from [29, Proposition 1.13], because continuous convergence implies assumptions $(A7)$–$(A8)$ therein.

4.2 Stability finite-horizon dynamic programming models

As another application, we perturb one of the simplest dynamic programming models, namely, the finite-horizon discrete discount model under certainty. Dynamic programming models have been widely used by a number of authors in various well-known papers on economic theory as Arrow et al. [3], Brock and Mirman [11], Kydland and Prescott [21], Lucas [27], and Lucas and Prescott [28], among others. Discrete dynamic programming models are frequently useful to address some discrete optimal control problems as done by Guigue et al. [16], Ha et al. [18], Murray and Yakowitz [30], and Ramadge and Wonham [32], among others.

We study the stability of a discrete time discounted dynamic programming model. To be more precise, of a finite-horizon version of the dynamic programming model under certainty as in Stokey and Lucas [35]. The classical model is stated as follows:

Given $x \in \mathbb{R}$, find $\{x^*_k\}_{k=0}^T \in \mathbb{R}^{T+1}$ such that

$$\sum_{k=0}^{T-1} \beta^k v(x^*_k, x^*_{k+1}) = \sup_{\{x_k\}_{k=0}^T \in \mathbb{R}^{T+1}} \sum_{k=0}^{T-1} \beta^k v(x_k, x_{k+1}),$$

subject to $x_0 = x$ and $x_{k+1} \in \Gamma(x_k)$, for all $k \in \{0, \ldots, T-1\}$.

Here, $\beta \in (0, 1]$ is the discount rate, $v\colon \mathbb{R}^2 \to \mathbb{R}$ is a bounded function, and $\Gamma\colon \mathbb{R} \rightrightarrows \mathbb{R}$ is a multifunction. Let us define the mapping

$$U[x, \Gamma] := \left\{\{x_k\}_{k=0}^T \in \mathbb{R}^{T+1} \colon x_0 = x \text{ and } x_{k+1} \in \Gamma(x_k), \forall k \in \{0, \ldots, T-1\}\right\}.$$

Berge's theorem allows us to prove that the function $\mu\colon \mathbb{R} \to \mathbb{R}$, defined by

$$\mu(x) := \sup_{\{x_k\}_{k=0}^T \in U[x,\Gamma]} \sum_{k=0}^{T} \beta^k v(x_k, x_{k+1}),$$

is continuous when v is continuous, Γ is nonempty-valued, continuous and locally bounded. Additionally, the following Bellman equation holds:

$$\mu(x) = \sup_{y\in\Gamma(x)} \{v(x,y) + \beta\mu(y)\}.$$

Moreover, the solution multifunction $\Lambda\colon \mathbb{R} \rightrightarrows \mathbb{R}^{T+1}$, defined by

$$\Lambda(x) := \left\{ \{x_k\}_{k=0}^T \in U[x,\Gamma] \colon \mu(x) = \sum_{k=0}^{T-1} \beta^k v(x_k, x_{k+1}) \right\},$$

is nonempty-valued, osc and locally bounded (cf. Remark 5).

The perturbed model is defined as follows. Let $\Gamma^n\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be multifunctions, $v^n\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be bounded functions and $\mu^n\colon \mathbb{R} \to \mathbb{R}^n$ be functions for all $n \in \mathbb{N}$ defined as the solution to

$$\mu^n(x) = \sup_{y\in\Gamma^n(x)} \{v^n(x,y) + \beta\mu^n(y)\}.$$

Also, for each $n \in \mathbb{N}$, we define the solution multifunction $\Lambda^n\colon \mathbb{R} \to \mathbb{R}^{T+1}$ by

$$\Lambda^n(x) := \left\{ \{x_k\}_{k=0}^T \in U[x,\Gamma^n] \colon \mu^n(x) = \sum_{k=0}^{T-1} \beta^k v^n(x_k, x_{k+1}) \right\}.$$

Before stating the main result of this section, we need two lemmas.

Lemma 41. *Let $\Gamma_1^n, \Gamma_2^n\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be multifunctions, for all $n \in \mathbb{N}$. If $\Gamma_1^n \xrightarrow{c} \Gamma_1$, with $\{\Gamma_1^n\}$ elb and $\Gamma_2^n \xrightarrow{c} \Gamma_2$, then $\Gamma_2^n \circ \Gamma_1^n \xrightarrow{c} \Gamma_2 \circ \Gamma_1$.*

Proof. Let $x^n \to x$. If $z \in \limsup_n (\Gamma_2^n \circ \Gamma_1^n)(x^n)$, then there exists $z^{n_k} \in \Gamma_2^{n_k}(y^{n_k})$ and $y^{n_k} \in \Gamma_1^{n_k}(x^{n_k})$ for every k such that $z^{n_k} \to z$. By hypothesis there exists a subsequence $\{y^{n_{k_j}}\}$ of $\{y^{n_k}\}$ converging to some $y \in \mathbb{R}$. As $\Gamma_2^{n_{k_j}}(y^{n_{k_j}}) \to \Gamma_2(y)$ (cf. Remark 10(4)) and $z \in \limsup_j \Gamma_2^{n_{k_j}}(y^{n_{k_j}})$, we infer that $z \in \Gamma_2(y)$. Since $y \in \limsup_k \Gamma_1^{n_k}(x^{n_k}) \subset \Gamma_1(x)$, we have $z \in \Gamma_2(\Gamma_1(x))$. Hence $\limsup_n (\Gamma_2^n \circ \Gamma_1^n)(x^n) \subset (\Gamma_2 \circ \Gamma_1)(x)$.

If $z \in (\Gamma_2 \circ \Gamma_1)(x)$, then $z \in \Gamma_2(y)$ for some $y \in \Gamma_1(x)$. As $\Gamma_1(x) \subset \liminf_n \Gamma_1^n(x^n)$, there exists $y^n \in \Gamma_1^n(x^n) \to y$. Since $\Gamma_2(y) \subset \liminf_n \Gamma_2^n(y^n) \subset \liminf_n \Gamma_2^n(\Gamma_1^n(x^n))$, we conclude that $z \in \liminf_n \Gamma_2^n(\Gamma_1^n(x^n))$. Hence $(\Gamma_2 \circ \Gamma_1)(x) \subset \liminf_n (\Gamma_2^n \circ \Gamma_1^n)(x^n)$. □

Lemma 42. *Let $K_k^n, K_k\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be given multifunctions for $k \in \{1,\dots,T\}$ and $\Phi^n, \Phi\colon \mathbb{R} \rightrightarrows \mathbb{R}^T$ be multifunctions defined by $\Phi^n(x) := \prod_{k=1}^T K_k^n(x)$ for every $n \in \mathbb{N}$ and $\Phi(x) := \prod_{k=1}^T K_k(x)$. If $K_k^n \xrightarrow{c} K_k$ for every $k \in \{1,\dots,T\}$, then $\Phi^n \xrightarrow{c} \Phi$.*

Proof. It directly follows by the convergence of factors in the cartesian product. □

Theorem 43. *Let $v^n \xrightarrow{c} v$ and $\Gamma^n \xrightarrow{c} \Gamma$. Then*

(a) $\mu^n \xrightarrow{c} \mu$.

(*b*) $\limsup_n \Lambda^n(x^n) \subset \Lambda(x)$ *for every* $x^n \to x$ *with* Λ *compact-valued.*

(*c*) $\{\Lambda^n\}$ *is elb.*

(*d*) $\Lambda(x) = \bigcap_{\epsilon>0} \liminf_n \Lambda^{n,\epsilon}(x^n)$ *for every* $x^n \to x$, *where*

$$\Lambda^{n,\epsilon}(x) = \left\{ \{x_k\} \subset U[x, \Gamma^n] : x_0 = x, \mu^n(x) < \sum_{k=0}^{T-1} \beta^k v^n(x_k, x_{k+1}) + \epsilon \right\}.$$

Proof. Let $u^n \colon \mathbb{R}^{T+1} \to \mathbb{R}$ be defined by $u^n(x_0, x_1, \ldots, x_T) := \sum_{k=0}^{T-1} \beta^k v^n(x_k, x_{k+1})$. As $v^n \xrightarrow{c} v$, we have $u^n \xrightarrow{c} u$. Let $\Phi^n \colon \mathbb{R} \to \mathbb{R}^T$ be the multifunction defined by $\Phi^n(x) := \{x\} \times \prod_{k=1}^{T-1} \Gamma^{n,(k)}(x)$, for each $n \in \mathbb{N}$, where $\Gamma^{n,(k)} := \Gamma^n \circ \cdots \circ \Gamma^n$ (k times). By Lemmas 41 and 42, we have $\Phi^n \xrightarrow{c} \Phi$, where $\Phi(x) := \{x\} \times \prod_{k=1}^{T-1} \Gamma^{(k)}(x)$. By Corollary 24, the proof is complete. □

The model presented in this work is similar to the discrete discounted dynamic programming model with a discontinuous value function introduced by Ausubel and Deneckere in [7]. In this case, our convergence methods do not apply since we deal with continuous value functions.

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